NEW TRENDS IN NUMERICAL SIMULATION OF THE MOTION OF SMALL BODIES OF THE SOLAR SYSTEM

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Abstract. A brief survey of the results obtained by the authors in the development and investigation of the algorithms of numerical simulation of the motion of solar system small bodies is given. New approaches to the construction of the algorithms of high-accuracy numerical simulation of the dynamics of small bodies and the methods of the determination of the domain of their possible motions are presented.

Key words: solar system small bodies, dynamics, numerical simulation

1. Introduction

Solar system small bodies such as asteroids, comets and satellites are characterized by a wide variety of orbits and complex structure of perturbations. For this reason numerical simulation of the motion of these objects is connected with some difficulties. In the paper we present a set of new approaches allowing a way over them.

We shall consider the algorithms meant for high-accuracy long-term numerical simulation of the motion of the small bodies and the algorithms of the determination of the domains of their possible motions.

Continuing our earlier researches (Bordovitsyna, 1984; Bordovitsyna and Sharkovsky, 1994) for the application of regularizing and stabilizing transformations in the algorithms of high-accuracy numerical simulation of the celestial bodies motion we have constructed new Encke-type algorithms in Kustaanheimo–Stiefel (KS) variables (Stiefel and Scheifele, 1971). These algorithms and the analysis of the results of their numerical realization are presented in Section 2.

The problem of numerical simulation of close encounters of small bodies with large planets is discussed in Section 3. New approaches for the determination of the domains of possible motions of small bodies are given in Section 4.

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2. Algorithms of High-accuracy Numerical Simulation

2.1. EQUATIONS OF MOTION AND THEIR PECULIARITIES

The motion of a material particle with the mass m in the gravitational field of the central body with the mass M under the action of perturbed forces can be written as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mu x}{r^3} = -\frac{\partial V}{\partial x} + F, \tag{1}$$

with initial conditions

$$x_0 = x(t_0), \qquad \dot{x}_0 = \dot{x}(t_0). \tag{2}$$

Here $x = (x_1, x_2, x_3)^T$ is the position vector, t is physical time, $r = |x|$, $\mu = k^2(M + m)$, moreover m is infinitely small by comparison with M, k^2 is the universal gravitational constant, $V = V(x,t)$ is a perturbed function of potential forces and F is the vector of the accelerations due to the forces which have no potential.

As it is well known, Equations (1) are singular in the vicinity of the central mass. For orbits having large eccentricities the presence of the singularity at the origin of the coordinate frame causes strong and nonuniform changes of the right-side functions of the motion equations. The same thing takes place when the small bodies approach some planets. In the process of numerical integration these nonuniformities require a regular change of integrating step size. It involves losses in accuracy of numerical solution and wasteful expenditure of computer time.

Besides, the solutions of Equations (1) are unstable in the Lyapunov sense even in the case of Keplerian motion. And this instability intensifies the influence of errors generated in the numerical process.

There are different ways of eliminating the losses in the efficiency of numerical integrating the equations of the motion of Solar system small bodies. These are the use of a computer word of large length and high order numerical methods as well as the transformations allowing completely or partly to avoid the singularity mentioned above.

2.2. USING REGULARIZING AND STABILIZING TRANSFORMATIONS

Kustaanheimo–Stiefel transformations (Stiefel and Scheifele, 1971):

$$x = L(u)u \quad \text{and} \quad 2\omega \mathrm{d}t = r\mathrm{d}E,$$

reduce the motion Equation (1) to the perturbed harmonic oscillator

$$\frac{\mathrm{d}^2 u}{\mathrm{d}E^2} + \frac{1}{4}u = \frac{r}{8\omega^2} L^T(u)\left(-\frac{\partial V}{\partial x} + F\right) - \frac{Vu}{8\omega^2} - \frac{1}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}E}\frac{\mathrm{d}u}{\mathrm{d}E}, \tag{3}$$

and the time equation

$$\frac{\mathrm{d}t}{\mathrm{d}E} = \frac{r}{2\omega}, \tag{4}$$

where the frequency ω is given by the equation

$$\frac{\mathrm{d}\omega}{\mathrm{d}E} = -\frac{r}{8\omega^2}\left(\frac{\partial V}{\partial t} + (\dot{x}, F)\right). \tag{5}$$

Here u is the 4-dimensional vector in KS-space, $L(u)$ is the well-known KS-matrix

$$L(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix},$$

and E is the so-called generalized eccentric anomaly as a new independent variable.

Equations (3)–(5) and their solutions are completely regular in the vicinity of the central body.

When perturbations are absent the solution of (3)–(5) has the form

$$\begin{aligned} u &= \alpha\cos\frac{E}{2} + \beta\sin\frac{E}{2}, \qquad \omega = \omega_0 = \text{const}, \\ t &= \tau - \frac{1}{\omega}\left(u, \frac{\mathrm{d}u}{\mathrm{d}E}\right), \end{aligned} \tag{6}$$

where α, β are regular elements, 4-dimensional vector constants, and the time element τ is a linear function of E.

Corresponding to (3)–(5) and (6) the equations of perturbed motion in the elements $q = (\alpha, \beta, \omega)^T$ and τ can be written as

$$\frac{\mathrm{d}q}{\mathrm{d}E} = R\left(\frac{E}{2}\right)Q, \tag{7}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}E} = \frac{1}{8\omega^3}\left[\mu - 2rV + r\left(x, F - \frac{\partial V}{\partial x}\right)\right] - \frac{2}{\omega^2}\frac{\mathrm{d}\omega}{\mathrm{d}E}\left(u, \frac{\mathrm{d}u}{\mathrm{d}E}\right), \tag{8}$$

where

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

$Q = (Q_1, Q_2, Q_3)^T$ is the vectorial perturbing function whose components Q_1, Q_2 in turn are the 4-dimensional vectors but Q_3 is the scalar:

$$Q_1 = 0, \qquad Q_2 = \frac{r}{4\omega^2}L^T(u)\left(-\frac{\partial V}{\partial x} + F\right) - \frac{Vu}{4\omega^2} - \frac{2}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}E}\frac{\mathrm{d}u}{\mathrm{d}E}, \tag{10}$$

$$Q_3 = -\frac{r}{8\omega^2}\left(\frac{\partial V}{\partial t} + (\dot{x}, F)\right). \tag{11}$$

The introduction of the time element τ, which varies almost linearly in weak perturbed motion, allows us to slacken the sensitivity of the equation of a quick secular variable for large values of the eccentricity. Thus the whole system of equations is completely insensitive to large values of the orbital eccentricity.

KS-theory in combination with high-order numerical methods is a powerful means for solving a lot of problems in celestial mechanics. Two examples below demonstrate some advantages of the stabilizing and regularizing transformations as applied to the problems of numerical simulation of asteroid and satellite motion.

In Figure 1 and 2 for the example of simulating the motion of the asteroid Phaethon the characteristics of accuracy and speed of the Everhart integrator (Everhart, 1974) are given when using the rectangular coordinates x and the KS-variables u. The plots in the figures show that the efficiency of the KS-algorithm is essentially higher than that of the classical algorithm. The high accuracy of the KS-algorithm can be achieved in weak conditions of the integrator and at the price of negligible losses in computer time. In principle the same accuracy can be achieved in numerical integration of the classical equations. However, it is repaid with low speed of calculations. Moreover, it should be noted, the efficiency region for the classical algorithm is significantly narrower.

In Figure 3 and 4 the same characteristics are given for the satellite of Jupiter, Metis. Judging from the plots the efficiency of KS-algorithm is also higher. We gave these examples to emphasize once more that the application of regularizing and stabilizing transformations to constructing high-efficient algorithms for simulating the motion of solar system small bodies deserves special attention.

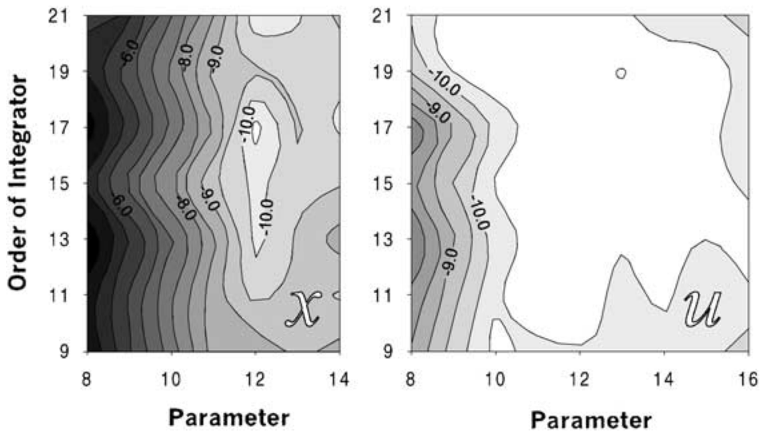

Figure 1. The order of the error Δr obtained by the forward-and-backward integration method for different conditions of integration. Asteroid Phaethon, 1000 rev.

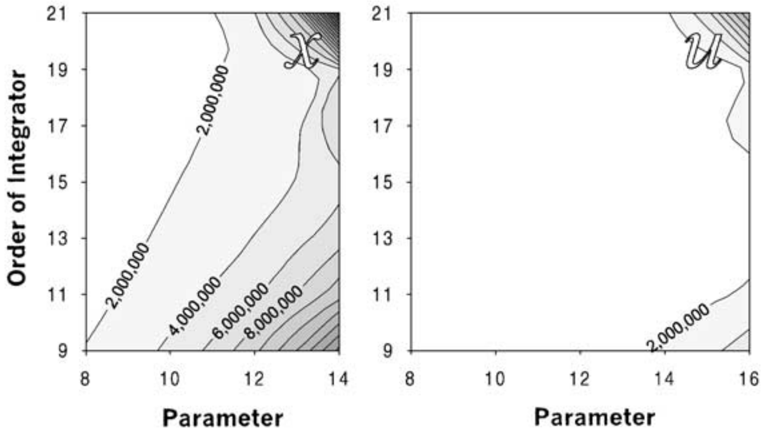

Figure 2. The speed in numbers of calling the subroutine of the perturbation function NCF for different conditions of integration. Asteroid Phaethon, 1000 rev.

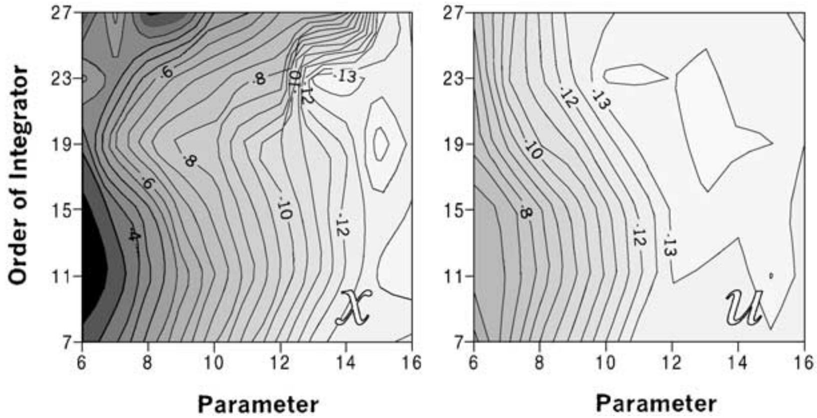

Figure 3. The order of the error Δr obtained by the forward-and-backward integration method for different conditions of integration. Satellite Metis, 1000 rev.

2.3. ENCKE-TYPE ALGORITHMS IN KS-VARIABLES

An essential defect in the mentioned KS-systems is that they contain the differential equations of quick variables, moreover some of them such as t and τ increase unboundedly. As is well known, the equations of quick variables are characterized by large values and complex variations of the right-side members. Therefore their numerical integration runs by small steps and with low accuracy.

The Encke method (Encke, 1852) allows this problem to be solved. It reduces the right-side members of the differential equations by using the deflections (perturbations) of real coordinates from an intermediate orbit as new integrated variables.

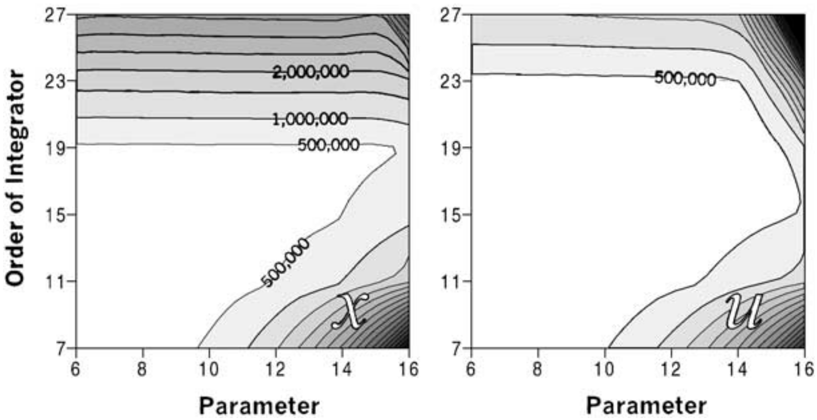

Figure 4. The speed in numbers of calling the subroutine of the perturbation function NCF for different conditions of integration. Satellite Metis, 1000 rev.

2.3.1. *Classical approach*

Let us take the Keplerian orbit in KS-space as an intermediate one. In accordance with (3), (5) and (8) it is described by the equations

$$\frac{\mathrm{d}^2 u_\mathrm{K}}{\mathrm{d}E^2} + \frac{1}{4} u_\mathrm{K} = 0, \qquad \frac{\mathrm{d}\omega_\mathrm{K}}{\mathrm{d}E} = 0, \qquad \frac{\mathrm{d}\tau_\mathrm{K}}{\mathrm{d}E} = \frac{\mu}{8\omega_\mathrm{K}^3},$$

which have the simple analytical solutions

$$\begin{aligned} u_\mathrm{K} &= \alpha_\mathrm{K}\cos\frac{E}{2} + \beta_\mathrm{K}\sin\frac{E}{2}, \qquad \alpha_\mathrm{K} = \alpha_0, \qquad \beta_\mathrm{K} = \beta_0, \\ \tau_\mathrm{K} &= \frac{\mu}{8\omega_\mathrm{K}^3} E + \tau_0, \qquad \omega_\mathrm{K} = \omega_0. \end{aligned} \tag{12}$$

Here and below the indexes K and 0 denote Keplerian and initial variables respectively.

Then the Encke equations in the KS-interpretation turn into the form (Bordovitsyna et al., 1998a)

$$\frac{\mathrm{d}^2\delta u}{\mathrm{d}E^2} + \frac{1}{4}\delta u = \frac{r}{8\omega^2} L^T(u)\left(-\frac{\partial V}{\partial x} + F\right) - \frac{Vu}{8\omega^2} - \frac{1}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}E}\frac{\mathrm{d}u}{\mathrm{d}E}, \tag{13}$$

$$\begin{aligned} \frac{\mathrm{d}\delta\tau}{\mathrm{d}E} &= \frac{1}{8\omega^3}\left[\mu\left(1 - \frac{\omega^3}{\omega_\mathrm{K}^3}\right) - 2rV + r\left(x, F - \frac{\partial V}{\partial x}\right)\right] - \\ &\quad - \frac{2}{\omega^2}\frac{\mathrm{d}\omega}{\mathrm{d}E}\left(u, \frac{\mathrm{d}u}{\mathrm{d}E}\right), \end{aligned} \tag{14}$$

$$\frac{\mathrm{d}\delta\omega}{\mathrm{d}E} = -\frac{r}{8\omega^2}\left(\frac{\partial V}{\partial t} + (\dot{x}, F)\right), \tag{15}$$

$$\delta u = u - u_\mathrm{K}, \qquad \delta\tau = \tau - \tau_\mathrm{K}, \qquad \delta\omega = \omega - \omega_\mathrm{K}.$$

2.3.2. *A modified Encke-method*

An original approach to realize the Encke method was offered by Sharkovsky (1990). It is based on the use of Keplerian eccentric anomaly E_{K} as an independent variable and of some properties of the rotation matrices R (9).

Sharkovsky showed the elegance of his method in equations of the form

$$\frac{\mathrm{d}q}{\mathrm{d}E} = R\left(\frac{E}{2}\right)Q, \qquad \frac{\mathrm{d}t}{\mathrm{d}E} = \frac{r}{2\omega}.$$

The transition to the new independent variable is realized by means of the transformation

$$\mathrm{d}E = (1+\varepsilon)\mathrm{d}E_{\mathrm{K}}, \qquad \varepsilon = \frac{r_{\mathrm{K}}\omega}{\omega_{\mathrm{K}} r} - 1. \tag{16}$$

Then the equations of motion can be written as

$$\frac{\mathrm{d}q^*}{\mathrm{d}E_{\mathrm{K}}} + \frac{1}{2}\varepsilon I q^* = (1+\varepsilon)R\left(\frac{E_{\mathrm{K}}}{2}\right)Q, \tag{17}$$

$$\frac{\mathrm{d}t}{\mathrm{d}E_{\mathrm{K}}} = \frac{r_{\mathrm{K}}}{2\omega_{\mathrm{K}}}, \tag{18}$$

where

$$q^* = R^T\left(\frac{\delta E}{2}\right)q, \qquad I = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\delta E = E - E_{\mathrm{K}}$ is the correction due to perturbations in the eccentric anomaly.

The time Equation (18) has an analytical solution which, as a matter of fact, constitutes the generalized Keplerian equation

$$t - t_0 = \frac{\mu}{8\omega_{\mathrm{K}}^3}(E_{\mathrm{K}} - \sin E_{\mathrm{K}}) + r_0\left[\frac{\sin E_{\mathrm{K}}}{2\omega_{\mathrm{K}}} + \dot{r}_0\frac{1-\cos E_{\mathrm{K}}}{4\omega_{\mathrm{K}}^2}\right]. \tag{19}$$

Therefore the time equation can be replaced by (19) and thus, struck off the system solved numerically.

It is clear that in the case of small perturbations the application of the obtained system (17) is advantageous when ε is close to zero. However the growth and appreciable periodic oscillations of the value ε due to the influence of perturbations cause strong variations of the right-side functions of the equations and, in turn, result in the enlargement of numerical errors.

A few words should be mentioned on the problem of subtracting two almost equal terms in Equations (14) and (17) (Encke terms):

$$1 - \frac{\omega^3}{\omega_{\mathrm{K}}^3}, \qquad \varepsilon = \frac{r\omega_{\mathrm{K}}}{r_{\mathrm{K}}\omega} - 1.$$

The direct subtractions in Encke terms cause low accuracy of their calculation. However one can avoid this difficulty by employing the Encke transformation. It reduces the subtractions to the expressions

$$1-\frac{\omega^3}{\omega_{\mathrm{K}}^3}=-\frac{\delta\omega}{\omega_{\mathrm{K}}}\left[\left(\frac{\omega}{\omega_{\mathrm{K}}}\right)^2+\left(\frac{\omega}{\omega_{\mathrm{K}}}\right)+1\right],$$

$$\frac{r\omega_{\mathrm{K}}}{r_{\mathrm{K}}\omega}-1=\frac{\delta r}{r_{\mathrm{K}}}\frac{\omega_{\mathrm{K}}}{\omega}-\frac{\delta\omega}{\omega},$$

where

$$\delta r=(u_{\mathrm{K}}+u,\delta u),\qquad r_{\mathrm{K}}=(u_{\mathrm{K}},u_{\mathrm{K}}).$$

2.3.3. *New intermediate orbits*

Traditionally in the Encke method the orbit of the two-body problem is used as an intermediate one. There were attempts to improve the Encke method by using the intermediate orbits which include these or those forces affecting an investigated object.

There are a number of the Encke-type algorithms based on so-called superosculating orbits (Batrakov and Makarova, 1979; Batrakov and Mirmakhmudov, 1991; Shefer, 1998) where one uses the idea of the introduction of a fictitious gravitational center offered by Shaikh (1966).

Sorokin (1991) had exploited the Encke method based on the problem of two fixed gravitational centers. Borrowing Herrick's idea about the additional mass, (Herrick, 1972) Avdyuschev (1999) offers the KS-orbit taking partially into account the oblateness of a planet for approximating the quasi-circular equatorial motion of a planet's inner satellite.

The equation of Herrick's orbit has the form

$$\frac{\mathrm{d}^2x_J}{\mathrm{d}t^2}+\mu\left(1+\frac{3}{2}\frac{J_2b^2}{r_J^2}\right)\frac{x_J}{r_J^3}=0,$$

where J_2 is the factor of the second zonal harmonic in the expansion of the planetary potential and b is the equatorial radius of the planet. Here and below index J is used to denote the variables relating to the new intermediate orbit.

In KS-space the same orbit is described by the equations

$$\frac{\mathrm{d}^2u_J}{\mathrm{d}E^2}+\frac{1}{4}(1+4\Omega)u_J=0,\qquad \frac{\mathrm{d}\omega_J}{\mathrm{d}E}=0,$$

$$\frac{\mathrm{d}\tau_J}{\mathrm{d}E}=\frac{\mu}{8\omega_J^3}\left(1-\Omega\right),\qquad \Omega=\frac{(2\omega_J)^4J_2b^2}{2\mu^2}.$$

On the basis of the new intermediate orbit the Encke equations in KS-variables have been realized numerically and their efficiency has been examined in dynamical problems of some natural satellites.

2.4. ESTIMATION OF ALGORITHMIC EFFICIENCY

To estimate the efficiency of the algorithms, numerical experiment has been carried out as applied to simulating the motion of special asteroids and outer satellites.

The efficiency of an algorithm is understood as a feature defined by its accuracy and speed. The value of the maximum of the deflection $\Delta r = \sqrt{\Delta x^2}$ in the positional vector obtained by the well-known forward-and-backward integration method is taken as a measure of the algorithmic accuracy. An error in the position vector appearing as a result of inexact integrating the time equation is calculated as $\Delta r_t = \sqrt{\dot{x}^2}\Delta t$ and is added to Δr.

The number (NCF) of calling the subroutine of the right-side members of differential equations is used as a measure of calculation speed. The results of the experiment are given in Table I. Here $L = \lg \Delta r$ (lg AU) is the order of accuracy and NCF (10^5) is taken approximately. The analysis of the experimental results has allowed us to divide all the considered algorithms into three classes depending on efficiency.

Class I includes all the Encke-type algorithms. Class II consists of the algorithms written in regularizing and stabilizing variables. And the algorithm in rectangular coordinates is referred to as Class III.

The estimation of the efficiency of the algorithms in the problems of numerical simulation of the motion of inner satellites has been obtained for the example of three satellites: Phobos (I Mars), Amalthea (V Jupiter) and Mimas (I Saturn).

The orbits of all the objects are quasi-circular and quasi-equatorial. The motion of every satellite is simulated over the interval from the discovery of the satellite till the present time. The model of forces is limited by the influence from the gravitational field of an oblate planet and the Sun. For taking into account the oblateness of the central planet the factors J_2 were chosen: $1.9582 \cdot 10^{-3}$ for Mars at the mass

TABLE I

Efficiency classification of algorithms

		Class	L_{NCF}	
			$\Delta t = 100$ rev.	$\Delta t = 1000$ rev.
I	KS-Encke	(δu)	-14_1	-12_{10}
II	KS	(u)	-13_1	-11_{10}
III	Classical	(x)	$-13_{2.5}$	-11_{25}

TABLE II
Estimation of numerical integration errors

	Satellite	Ω, 10^{-4}	ΔT, rev.	$L(\delta u_J)$	$L(\delta u_K)$	$L(u)$	$L(x)$
J5	Amalthea	11	78900	−13	−11	−10	−3
S1	Mimas	9	81700	−13	−12	−11	−6
M1	Phobos	1	140400	−13	−13	−11	−6

$L = \lg \Delta r$ (lg AU).

1/3098710, $1.4736 \cdot 10^{-2}$ for Jupiter at the mass 1/1047.35 and $1.6298 \cdot 10^{-2}$ for Saturn at the mass 1/3498 while the mass of the Sun is 1.

The outcomes of the estimation given in the Table II demonstrate an advantage of KS-algorithms over the classical ones for all the objects investigated. Only the transition from x-space to u-space makes the integrating accuracy higher by 4–7 orders. The application of the equations in Encke form perfects the accuracy of KS-algorithms by not less than 2 orders.

Taking into account the effect of the planetary oblateness in intermediate orbits and decreasing perturbations in the Encke method in this way raise the integrating accuracy significantly, especially when the values of Ω are large (Amalthea, Mimas). It should be noted that the accuracy of the classical Encke method in KS-variables goes down when Ω increases. But the accuracy of the generalized one is quite insensitive to the variation of Ω.

In addition to the comments above, attention should also be paid to the fact that all these accuracy estimations are not absolute since they are obtained under similar algorithmic speed. For example, the accuracy of the classical algorithms could be raised by several orders but this accuracy would be achieved at the price of a considerable increase in computations.

3. Algorithms for Numerical Simulation of Close Encounters of Small Bodies with Planets

The problem of the numerical investigation of close encounters of small bodies with large planets is very difficult. The point is that the equations describing the dynamics during close encounters have singularities in the vicinities of planets. The KS-transformations consider above eliminate only the one singularity in the central body while the other ones remain. Therefore Stiefel–Scheifele KS-theory does not solve the problem of close encounters.

The graph on Figure 5 demonstrates the efficiency in accuracy of the Encke algorithm $L(\Delta\tau_E)$ in KS-elements $(\delta q, \delta\tau)$ by comparison with the original one (Δr) in the same elements (q, τ) for numerical simulations of the motion of

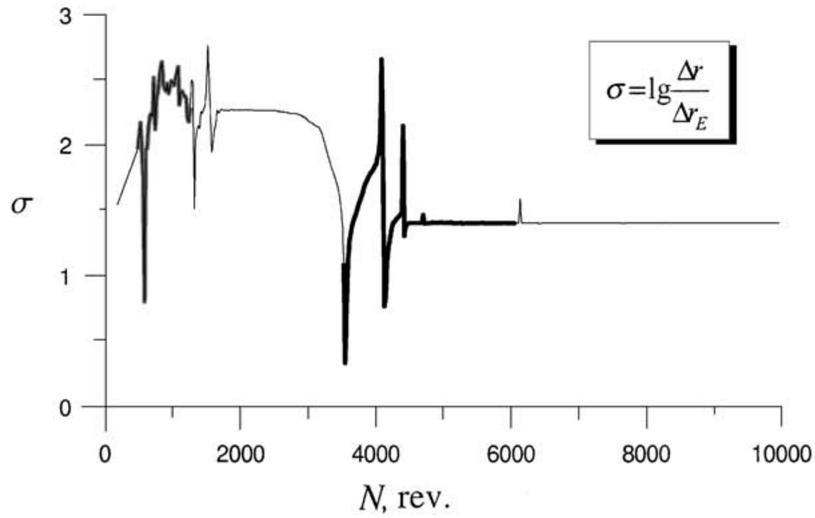

Figure 5. Advantage of Encke algorithm.

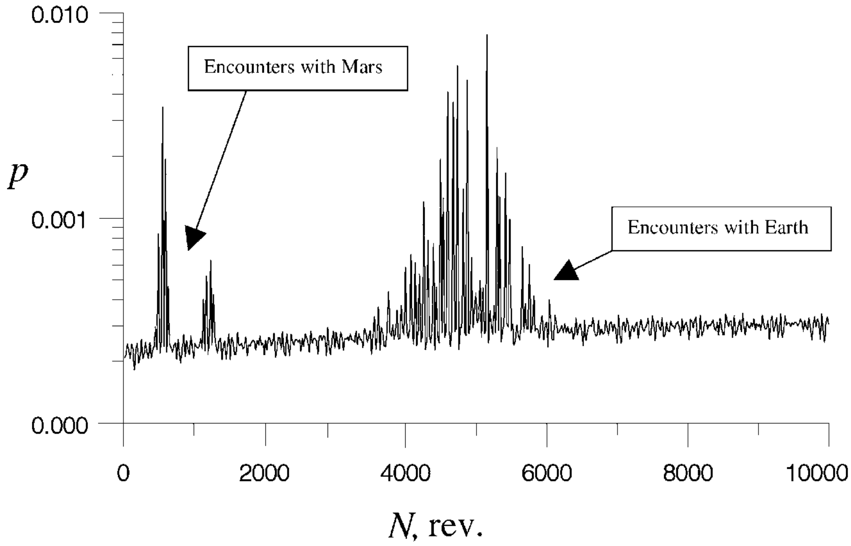

Figure 6. Relative perturbations.

asteroid Phaethon over time interval of 10 000 revolutions of the object. The part of the curve in Figure 5 where the loss in accuracy takes place corresponds directly to the part of the curve on Figure 6 where strong perturbations, caused by close encounters with the Mars and the Earth, occur. Here p is the ratio of the total perturbation value to the value of the central force.

The use of the double KS-transformation (Aarseth and Zare, 1974) is effective only when a small body comes deep within the action sphere of a planet (Shefer, 1990).

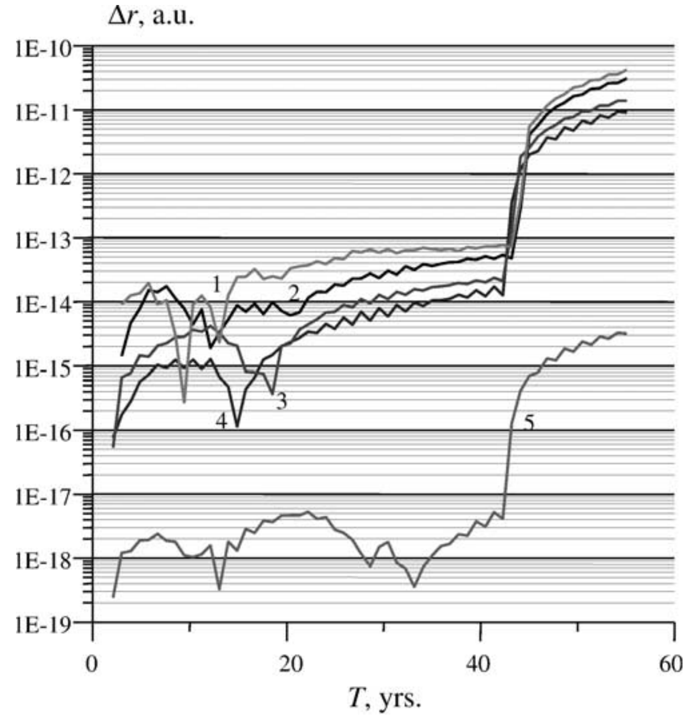

Figure 7. Estimation of the accuracy in positional vector when using different algorithms of the prediction of motion. Asteroid 1991 VG.

The drastic means of solving the problem consist in the use of high-precision computer word. What it gives is shown on Figure 7. Here the results of numerical investigation the asteroid dynamics during close encounters by using a high-precision computer word are presented. As an investigated object the asteroid 1991 VG is taken. Over the considered time interval it has had a lot of close encounters with the Earth, one of them is 0.0031 a.u. in December, 21, 1991.

Together with the classical equations of motion (1) denoted below as (x_t) we have used the equations (x_s) (Bordovitsyna et al., 1998b) with the new independent variable s connected with the time t by the differential relation

$$U\,\mathrm{d}t = \mathrm{d}s, \quad \text{where} \quad U = \sum \mu_i/\Delta_i,$$

in which μ_i is the gravitational parameter of massive body i including the Sun and Δ_i is its distance to the small body.

On Figure 7 the estimations of the accuracy in position vector for various precisions of computer word determined in decimal digits L for various orders N of Gragg-Richardson numerical method (Hairer et al., 1987) and for two forms of differential equations are given in Table III.

The estimations show that the loss of accuracy due to close encounters takes place in all cases. However, the use of a large scale computer word and a numer-

TABLE III
Parameters of simulation

Curve	L	N	Equations
1	19	18	x_t
2	19	18	x_s
3	25	18	x_t
4	25	18	x_s
5	28	26	x_s

ical high order method with the regularized differential equations (x_s) allow us to put errors to insignificant digits. At the same time the speed of the regularizing algorithm is essentially higher than that of the classical algorithm.

4. Algorithms for Determination of Domain of Possible Motions

4.1. SETTING OF PROBLEM

The accuracy of numerical simulation of the motion of Solar system small bodies is determined not only by the accuracy of numerical methods but also by the accuracy of assigning the initial parameters of the simulation. The initial parameters of the motion of real objects are defined by the well-known least squares method (LSM) or other methods by the observations which have errors.

So, in constructing the numerical models of the motion of a real object, it may be more rightful to assign its initial orbital elements in the form of a certain domain of their possible values and to regard the dynamical evolution of the object as an evolution of the domain. Lately that approach is getting more popular especially as there exist problems (for instance, the problems of the identification of objects) where this approach is, in general, the only possible one. We shall consider the problem of numerical simulating motion in the following setting.

The equations of motion can be written in the form

$$\dot{q} = f(q, C), \tag{20}$$

where $q(t)$ is the m-dimensional vector of dynamical variables, any of the considered ones above in Section 2, C is the known vector of constants of the model. The initial conditions $q_0 = q(t_0)$ are in the domain of possible motions R_0, that is $q_0 \in R_0$.

4.2. ALGORITHMS FOR DETERMINATION OF INITIAL DOMAINS OF POSSIBLE MOTIONS

There exist several ways to build the initial domains of motions R_0.

In the classical way, when it is expected that the law of distribution of errors of observations is close to normal one, initial domains of possible motions are defined by LSM-evaluations of the vector of initial parameters $\hat{q}_0$ and by the covariance matrix of their errors $\hat{D}_0$,

$$R_0 \colon N\left(\hat{q}_0, k^2\hat{D}_0\right), \qquad k = 1, 2, 3, \tag{21}$$

where k is the gain factor of LSM-evaluations of the covariance matrix of errors in initial parameters. In the case, when the law of distribution of errors of observations greatly differs from the normal one, one has to search other ways to assign initial domains of object motions.

Chernitsov has suggested using in the correlation (21) the gain factor $k > 3$ and has made an attempt to determine the values experimentally of this factor for different objects and different conditions of their observability.

Miunonen (1996) has suggested in bad-conditioned problems to use eigen-vectors for the construction of an initial bunch of trajectories, choosing randomly along with them the parameters of initial orbits within three sigmas. Under well conditioned problems this way reduces to the classical one.

An original way of assigning the initial domains of possible motions was offered by Milani et al. (2000)

$$R_0 \colon \Delta q^T C_q \Delta q \leqslant \sigma^2, \qquad \sigma > \sigma_0\sqrt{n-m}. \tag{22}$$

Here Δq is the m-dimensional vector defined by $q - q_0$ where q_0 is the solution best fitting the available observations, C_q is the normal matrix.

Under that assignment of the domain of initial values by its boundary is the ellipsoid defined by the inequality (22). Milani et al. have offered an approximate method of finding boundary points of the domain of possible motions.

Developing the idea of Milani et al. Chernitsov suggested an iterative algorithm for determining the borders of the domain of initial values, having presented it in the form of the ensembles of points which are a solution of the nonlinear equation of the type

$$\Psi(q_0, \sigma) = \Phi(q_0) - \sigma^2 = \Delta l^T(q_0) P \Delta l(q_0) - \sigma^2 = 0. \tag{23}$$

Here $\Delta l(q_0)$ is a n-dimensional vector of residuals $(O - C)$, P is a weight matrix, σ is the parameter determining the initial domain containing the true motion in the phase space, σ_0 is the mean error. Under this assign of the domain of initial values by its boundary is a surface of constant level of the minimized function $\Phi(q_0)$ by the method of least squares. A peculiarity of Equation (23) is the multiplicity of solutions. However, using a generalized analogue of Newton's method allows us to define boundary points for the domain of possible motions exactly.

TABLE IV

Determination of boundaries of initial domains of motion from observations of one appearance. Asteroid Icarus

t, year	ΔT, day.	N	σ_0	$\Delta\sigma$	k	$\sigma_{\max} - \sigma_{\min}$
1949	17	5	$0''.8$	$1''.2$	7	$0''.518$
1953	14	11	1.9	0.1	2	0.134
1954	24	4	1.4	0.6	2	1515.000
1958	54	8	0.5	0.3	4	0.001
1965	30	10	0.9	0.4	4	0.562
1966	29	8	1.0	0.1	2	0.335
1968	65	17	0.5	0.1	4	0.008
		496	0.5	1.1	12	0.008
1976	30	6	0.4	0.9	8	0.011
1977	11	3	1.4	1.4	3	1.687
1986	38	8	0.7	0.4	4	0.002
1987	73	46	1.2	0.1	3	0.002
1992	37	5	0.3	0.8	6	0.211

The complication of the problem of constructing initial domains of possible motions is demonstrated by the data presented in Tables IV and V.

The estimations have been obtained for the examples of numerical simulation of the motion of asteroids Icarus and Toutatis for different samples of observability. In the capacity of the samples we have taken the observations of various appearances of the asteroid. Initial domains of possible motions have been calculated by LSM-evaluations of the vector of the initial parameters $\hat{q}_0$ and by the covariance matrix of their errors $\hat{D}_0$. The orbital parameters calculated for the observations of all the considered appearances of Icarus and Toutatis have been chosen as 'true' ones. The gain factor k shows how the elements of the covariance matrix $\hat{D}_0$ have to be changed so that the 'true' motion should belong to the surface of the ellipsoid

TABLE V

Determination of boundaries of initial domains of motion from observations of one appearance. Asteroid Toutatis

t, year	ΔT, day	N	σ_0	$\Delta\sigma$	k	$\sigma_{\max} - \sigma_{\min}$
1988–89	267	145	$1''.43$	$0''.03$	4	$0''.001$
1992–93	282	347	0.890	0.004	3	0.005
1995–97	727	218	0.67	0.03	0.4	3.323

enveloping the surface of constant level of the minimized function $\Phi(q_0)$ of the method of least squares.

In the first columns of the tables the year of a considered appearance is indicated. In the second and third columns we have the time interval ΔT covered by the observations and the number N of a considered appearance respectively. In the fourth columns σ_0 is the mean error of fitting observations of LSM estimations obtained for observations of considered appearance. In the fifth columns we have the difference $\Delta\sigma$ among mean errors of fitting observations of a considered appearance of 'true' orbital parameters and the parameters obtained for this appearance. In the sixth columns the quantities of the gain factor k are given. In the last columns of the tables the maximal differences of the mean errors of fitting by observations in a considered appearance for the orbital parameters calculated in different points of the ellipsoid mentioned above are presented.

Analysis of the results presented in Table IV and V shows that the problem of constructing the initial domains of possible motion is really complicated. In some cases the quantities of the gain factor k are more than 3. It is evident that LSM-estimations of covariance matrices of errors of initial parameters give apocryphal submissive estimations of initial domains of possible motions. In the set of cases the presentations of the surface of constant level of the minimized function surfaces of the ellipsoids contained the estimations of 'true' orbital parameters are insufficiently precise.

4.3. ALGORITHMS FOR DETERMINING EVOLUTIONS OF THE DOMAINS OF POSSIBLE MOTIONS

The system of differential Equation (20) defines an evolution of possible trajectories of motion

$$q(q_0, t), \qquad q_0 \in N(\hat{q}_0, \hat{k}^2\hat{q}_0),$$

and, in particular, an evolution of the reference trajectory $q(\hat{q}_0, t)$. The traditional way of the evaluation of the domains of possible motions of the system determined by Equation (20) is the linear one, however more and more authors are biased to the opinion that this way does not give the adequate description of possible motions of the investigated object.

At the nonlinear approach the operation of displaying is realized by forming a sufficiently thick ensemble of possible trajectories $q(\hat{q}_0^i, t)$, beginning from the initial domain:

$$q_0^i \in N(\hat{q}_0, \hat{k}^2\hat{D}_0), \qquad i = 1\ldots, s.$$

Initial points of the trajectory q_0^i, as well as the initial domain $N(\hat{q}_0, \hat{k}^2\hat{D}_0)$ are determined by simulating the nondegenerated m-dimensional normal vector (Ayvazyan et al., 1983)

$$q^i(t_0) = A\eta^T + \hat{q}_0. \tag{24}$$

Here A is the triangular matrix such that

$$AA^T = \hat{k}^2 \hat{D}_0, \qquad \eta = (\eta_1, \dots, \eta_m),$$

where $\eta_j (j = 1, \dots, m)$ are independent (0, 1)-normally distributed random numbers; $i = 1, \dots, s$, where s is an amount of the initial points $q^i(t_0)$, determined by the length of simulated by random numbers generators sample of the vector η.

Our studies have shown that within the framework of model (20) for studying probabilistic evolution of small bodies motions it is sufficient to construct an ensemble from 500 up to 1500 trajectories.

Within the framework of the nonlinear approach it is easy to construct a map of the type

$$\{C_q\}_{t_0}^t \longrightarrow \{C_p\}_{t_0}^t \quad \text{and} \quad \{C_p\}_{t_0}^t \longrightarrow \{\hat{p}(t), \bar{D}_p(t)\}.$$

Here p is the vector of parameters different from the initial ones; t_0 and t are moments of time at which one formed the initial (start) domain $N(\hat{q}_0, \hat{k}^2 \hat{D}_0)$ and forecasted the domains $\{C_q\}_{t_0}^t$ and $\{C_p\}_{t_0}^t$, correspondingly; $\bar{D}_p(t)$ is a matrix characterizing the scatter of phase points of an ensemble of the trajectories $q(q_0^i, t)$ with regard to the point $q(\hat{q}_0, t)$ in the space of the parameters p.

Let us give several interesting examples.

On Figure 8 the divergence of the evaluations of the domains of possible motions of asteroid Toutatis obtained by the linear and nonlinear methods is shown. The main value of the nonlinear method is the fact that obtained on its base evaluations are much more profound and give a greater amount of interesting information on object's motion.

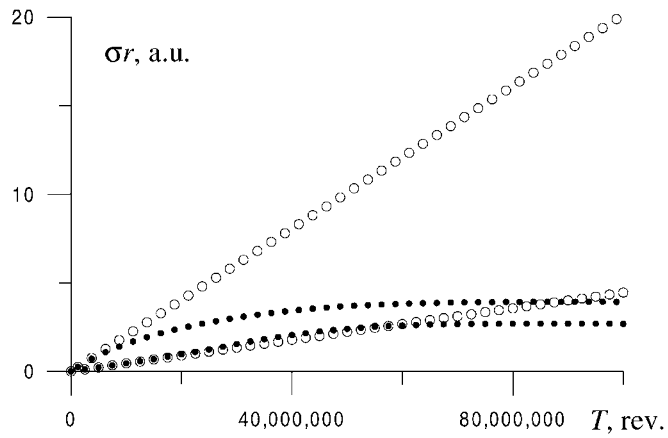

Figure 8. Comparison of nonlinear (•) and linear (○) astimations of accuracy of numerical simulating heliocentric positions of asteroid Toutatis in aphelia (lower) and perihelia (upper).

In Figure 9 the averaged evolution of the domain of possible motions of asteroid Toutatis for a time interval of 1,100 and 100 000 revolutions is shown. Initial conditions are determined from observations covering the time interval 1992–1993.

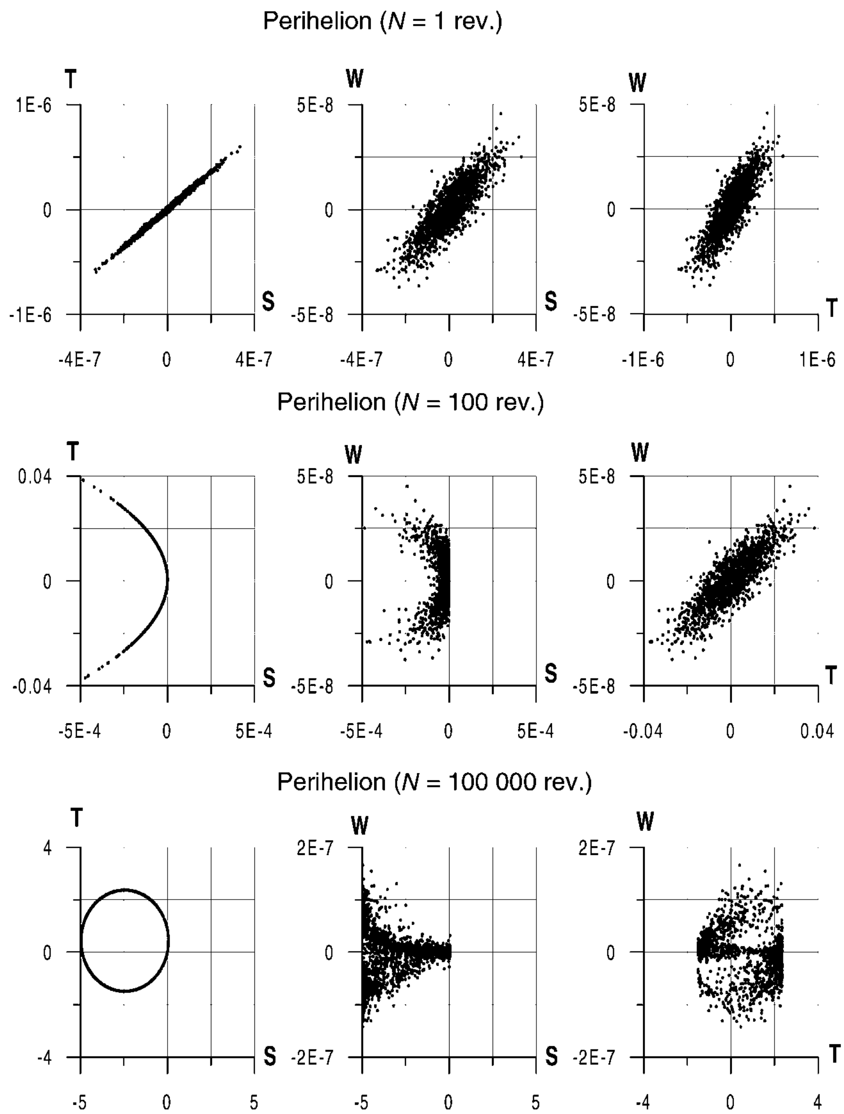

Figure 9. The evolution of the domain of of possible motions of asteroid Toutatis in projection on the orbital coordinate system planes. Unit of measurement is a.u.

From the given evaluations it is seen that the main deformation of the domain of possible motions comes in the course of time along the reference trajectory. Full uncertainty in calculated positions of an object comes after 100 000 revolutions, when the region of possible motions covers the whole trajectory.

In Figure 10 the evolution of the domain of possible motions of Toutatis is given, as has been stated, in detail for the time interval 1996–2016. Here Δr is the distance from a calculated object position to the 'true' position, σr is the calculated mean quadratic error. The 'true' motion is determined from observations covering

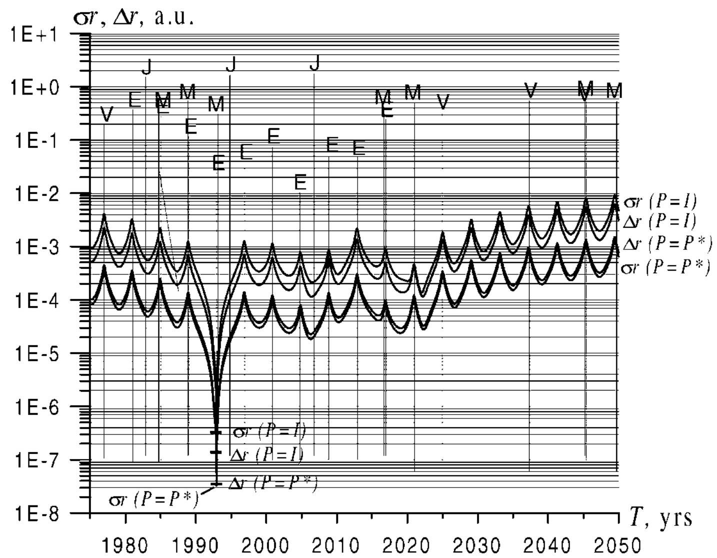

Figure 10. The evolution of the domain of possible motions of Toutatis in the time interval 1996–2016.

the time interval 1988–1993. The symbols V, E, M, J indicate the moments of encounters and the least distances to Venus, the Earth, Mars and Jupiter, respectively. The weight matrix P^* is calculated by all available observations. One can see that atypical evolutionary picture of changing the domain of possible motion of the asteroid and consequently the accuracy of forecast of its positions takes place revolution after revolution. On this interval of time expansions of the domain of possible motions do not occur but on the contrary under the action of perturbations even somewhat decrease. Stabilizing factors here are approaches and resonance with the Earth approximately 1 : 4 and with Jupiter 3 : 1.

5. Conclusions

Let us make conclusions as follows:

1. New Encke-type algorithms in Kustaanheimo–Stiefel variables have been presented. The algorithms do not contain the equations for quick variables and display high efficiency in numerical simulating the motion of special asteroids and planetary satellites. An intermediate KS-orbit taking partially into account the oblateness of a planet for approximating a quasi-circular equatorial motion of an inner satellite has been suggested as well. Using this orbit in the Encke method is most efficient when planetary oblateness is large.
2. The problem of numerical investigation of close encounters of small bodies with large planets have been analysised. It has been shown that the use of a

large scale computer word and a numerical high order method with the regularized differential equations allows to put the error of numerical simulation to insignificant digits.

3. New algorithms for determining initial domains of possible motions have been suggested. The analysis of using linear and nonlinear algorithms for determining evolutions of the domains of possible motions are presented. Several interesting numerical examples are given. These examples show that the main value of the nonlinear method is the fact that evaluations obtained on its base are much more profound and give a greater amount of interesting information on motion.

The results presented have been partially published in Russian astronomical periodicals but as a summary account like here are primarily given.

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