NUMERICAL STABILIZATION OF ORBITAL MOTION

Victor A. AVDYUSHEV
Celestial Mechanics and Astrometry Department, Applied Mathematics and Mechanics Institute, Tomsk, Russia, 634050, e-mail: sch@niipmm.tsu.ru

Abstract. Mainly, the author focuses on Baumgarte's method and its applications in satellite, asteroid, stellar and planetary problems. In the paper arguments are given for the use of energy relations for stabilization in the elliptical two-body problem. Stabilizing properties of Baumgarte's equations and others are discussed. A simple approach is proposed for stabilizing the equations of almost circular motion. By using Baumgarte's technique, the author derives stabilized equations of perturbed restricted three-body problem. It is shown experimentally that stabilization in the problems mentioned above can raise the accuracy of numerical integration by several orders.

Keywords: Lyapunov's instability, stabilization, numerical integration, orbital motion

1. Introduction

As is well-known, Lyapunov's instability of differential equations when integrated numerically is nutrient for cultivating various errors inevitably accompanying any numerical process. Errors on an integrating step become those in initial data on the next and are then increased by the instability step by step together with other errors implanted later on. Therefore, stable equations are preferable for numerical integration.

Weakening (or eliminating at best) the instability influence on the numerical solution is the main aim of the stabilization methods which we are going to discuss below.

J. Baumgarte and P. Nacozy are considered to be the founders of stabilization in celestial mechanics. Their approaches (Baumgarte, 1973; Nacozy, 1971) are based on the using of known integrals which are involved in numerical integration as a supplementary information on a solution of differential equations.

Technically, stabilization consists in correcting a numerical (erroneous) solution for its deviation from some integral surface in phase space of variables integrated. In a perturbed case, when the integral surface is dynamic and its integral parameter is variable, the same integral relation is used and, moreover, the set of equations is added with the equation of the integral parameter (Baumgarte, 1973; Bordovitsyna, Sukhoplyueva, 1980).

Stabilization is applied to either a numerical solution directly (Nacozy, 1971) or the set of equations by means of including the so-called stabilizing terms (Baumgarte, 1973). Obviously, the weaker the perturbations and the slower the integral parameter varies, the more effective the stabilization.

Baumgarte and Nacozy highlight energy relations of all for stabilization. In practice, these appear to be the best help in the struggle against Lyapunov's instability. That is the reason why the energy stabilization will be an object of our undivided attention.

Mainly, in the paper we purpose to examine the stabilization in some interesting problems of orbital dynamics and focuses on Baumgarte's method and its applications in satellite, asteroid, stellar and planetary problems.

2. Instability of Keplerian Motion

Let us consider two near Keplerian orbits $\mathbf{x}$ and $\mathbf{x}^{\sim}$ which differ from each other at an initial point of time t_0 by the small vectorial deviations $\Delta\mathbf{x}_0 = \mathbf{x}_0 - \mathbf{x}_0^{\sim}$ and $\Delta\dot{\mathbf{x}}_0 = \dot{\mathbf{x}}_0 - \dot{\mathbf{x}}_0^{\sim}$.

From two-body problem formulae it is not difficult to show that, up to first order small values, the deviation $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}^{\sim}$ in time t should satisfy the estimation

$$\Delta\mathbf{x} \sim \frac{3}{2}\frac{\Delta H_K}{H_K}\dot{\mathbf{x}}\Delta t. \tag{1}$$

Here $H_K = \text{const}$ is the Keplerian energy which is an integral of the orbit $\mathbf{x}$ (for specificity); $\Delta H_K = H_K - H_K^{\sim} = \text{const}$ is the difference of the orbital energies, and $\Delta t = t - t_0$.

Now if we take $\mathbf{x}^{\sim}$ for the exact orbit and $\mathbf{x}$ for the erroneous caused by the initial errors $\Delta\mathbf{x}_0$ and $\Delta\dot{\mathbf{x}}_0$ then we can make use of formula (1) for estimating the error $\Delta\mathbf{x}$ of the erroneous orbit at an arbitrary point of time.

Estimation (1) shows the evolution of $\Delta\mathbf{x}$ has a periodically secular character: the error increases towards the pericenter and decreases towards the apocenter together with the velocity $\dot{\mathbf{x}}$ but, generally, the error tends to grow. One also can see that the growth speed of $\Delta\mathbf{x}$ depends on the relative error in the Keplerian energy: the smaller $\Delta H_K/H_K$, the lower the growth speed of $\Delta\mathbf{x}$. But whichever small ΔH_K may be, sooner or later there come the point of time after which $\Delta\mathbf{x}$ gets large unacceptably. And it means that the Keplerian orbit is unstable in the sense of Lyapunov.

When the Keplerian orbit is integrated numerically, errors like $\Delta\mathbf{x}_0$ and $\Delta\dot{\mathbf{x}}_0$ arise on every integrating step over and over again. These, in turn, drift the numerical orbit off the integral surface specified by the energy $H_K^{\sim}$ in phase space of coordinates and their derivatives. The drift mainly reveals itself as a linear growth of ΔH_K and, according to (1), it results in a quadratic (superlinear) growth of $\Delta\mathbf{x}$.

Thus, in order to weaken the instability influence on the accuracy of numerical integration, it is necessary to correct the orbit when integrated laying it on the integral surface $H_K^{\sim}$ and suppressing the growth of ΔH_K.

3. Baumgarte's Method

A simple but powerful stabilizing method had been proposed by J. Baumgarte (1972; 1973). The basic idea of Baumgarte's method is to include the so-called stabilizing terms in the set of differential equations to compensate for the deviations of a numerical (erroneous) solution from some integral surface (of the exact solution). Here we consider the method applied to the set of first order differential equations.

Let the set

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{Q}(\mathbf{q}, t) \tag{2}$$

has the integral

$$C(\mathbf{q}) = \mathrm{const}, \tag{3}$$

where $\mathbf{q}$ is the vector of variables integrated, t is the independent variable, $\mathbf{Q}$ is the known vector-function of $\mathbf{q}$ and t.

We will stabilize the set by the reference integral surface determined by the algebraic equation

$$C(\mathbf{q}) = C^{\sim} \tag{4}$$

which is the requirement that some reference solution $\mathbf{q} = \mathbf{q}^{\sim}$ fulfils, that is to say $C(\mathbf{q}^{\sim}) \equiv C^{\sim}$.

Differentiating (4), we obtain

$$\frac{\mathrm{d}C^{\sim}}{\mathrm{d}t} = \frac{\partial C}{\partial \mathbf{q}} \cdot \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial C}{\partial \mathbf{q}} \cdot \mathbf{Q} \equiv 0. \tag{5}$$

Following Baumgarte, we include perturbing terms $\delta\mathbf{Q}$ in set (2) to carry any solution near by $\mathbf{q}^{\sim}$ to the reference integral surface so that the asymptotic stability by C is satisfied:

$$\Delta C \equiv C(\mathbf{q}) - C^{\sim} \to 0. \tag{6}$$

As a result, the stabilized set takes the form

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{Q} + \delta\mathbf{Q}. \tag{7}$$

Differentiating ΔC and taking into account (5) gives

$$\frac{\mathrm{d}\Delta C}{\mathrm{d}t} = \frac{\partial C}{\partial \mathbf{q}} \cdot \delta\mathbf{Q}. \tag{8}$$

Since on the integral surface $C(\mathbf{q}) = C^{\sim}$ the perturbing terms $\delta\mathbf{Q}$ must vanish, they are presented in the form

$$\delta\mathbf{Q} = \Delta C\mathbf{u}, \tag{9}$$

where $\mathbf{u}$ is an unknown vector-function yet which is to be found in accordance with (6).

Substituting (9) into (8) gives the differential equation of ΔC:

$$\frac{\mathrm{d}\Delta C}{\mathrm{d}t} = \Delta C\left(\frac{\partial C}{\partial \mathbf{q}} \cdot \mathbf{u}\right). \tag{10}$$

Let us define the function $\mathbf{u}$ as

$$\mathbf{u} \equiv -\gamma\frac{\partial C}{\partial \mathbf{q}}\left(\frac{\partial C}{\partial \mathbf{q}}\right)^{-2}, \tag{11}$$

where $\gamma = \mathrm{const} \neq 0$. Substituting (11) into (10) yields

$$\frac{\mathrm{d}\Delta C}{\mathrm{d}t} = -\gamma\Delta C. \tag{12}$$

Suppose that at an initial value t_0 of the independent variable the integral value $C_0 = C(\mathbf{q}_0)$ differ from the reference $C^{\sim}$ by ΔC_0. Then the solution of equation (12) is obtained in the form

$$|\Delta C| = |\Delta C_0|e^{-\gamma\Delta t}. \tag{13}$$

From this one can see the sign of γ determines whether the stabilization works: the parameter γ must be positive for ΔC to tend to zero when integrating forward ($\Delta t > 0$), otherwise, it must be negative when integrating backward ($\Delta t < 0$).

Finally, when the stabilizing terms have been determined, set (7) can be rewritten in the form

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{Q} - \gamma\Delta C\frac{\partial C}{\partial \mathbf{q}}\left(\frac{\partial C}{\partial \mathbf{q}}\right)^{-2}. \tag{14}$$

Let now set (2) be perturbed by some function $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{Q}(\mathbf{q}, t) + \mathbf{P}(\mathbf{q}, t). \tag{15}$$

In the perturbed case, the reference value $C^{\sim}$ of the integral function is variable. According to (4) and (5), its behaviour can be described by the differential equation

$$\frac{\mathrm{d}C^{\sim}}{\mathrm{d}t} = \frac{\partial C}{\partial \mathbf{q}} \cdot \mathbf{P}. \tag{16}$$

As in the unperturbed case, the stabilizing terms $\delta\mathbf{Q}$ (9) are also included in the set (15). Because $C^{\sim}$ is used for calculating $\delta\mathbf{Q}$, the stabilization implies adding equation (16) in the perturbed set (15). Using the manipulations stated above, it is not difficult to show the choice (11) of the function $\mathbf{u}$ is a guaranty for the stabilization to work.

As a result, the perturbed stabilized set is taken the form

$$\begin{aligned} \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} &= \mathbf{Q} + \mathbf{P} - \gamma \Delta C \frac{\partial C}{\partial \mathbf{q}} \left(\frac{\partial C}{\partial \mathbf{q}}\right)^{-2}, \\ \frac{\mathrm{d}C^{\sim}}{\mathrm{d}t} &= \frac{\partial C}{\partial \mathbf{q}} \cdot \mathbf{P}, \quad \Delta C = C(\mathbf{q}) - C^{\sim}. \end{aligned} \tag{17}$$

From the analytical point of view, set (17) is equivalent to set (15) at $\mathbf{q} = \mathbf{q}^{\sim}$ since $\Delta C \equiv 0$. However, the former is asymptotically stable by C whereas the latter is not. Such a stability of differential equations is very useful when they are integrated numerically because it allows one to hold their numerical solution nearby the integral surface and, in this way, to take into account some properties of the exact solution. While the integration of unstable equations is accompanied by the secular deviation of a numerical solution from the integral surface.

On the point concerning the initial conditions of the stabilized set, it should be mentioned only that the variable $C^{\sim}$ should start from the initial value of the integral function $C_0^{\sim} = C(\mathbf{q}_0)$.

In spite of the fact that stabilized differential equations are complicated and need more time than unstable ones to calculate their right-hand members, stabilization can essentially raise the speed of integration since it allows one to enlarge integrating steps keeping the accuracy of a numerical solution.

Nevertheless, the main practical difficulty in using stabilized sets consists in choosing of γ. The point is that one cannot give definite recommendations concerning this. Mathematically, according to (13) it seems the larger γ, the faster ΔC decreases to zero. But large γ can introduce stiffness in the equations of motion which the numerical method hardly "digests". As experiments show, the optimal γ to attain to the highest numerical accuracy is specific for each problem and it should be taken empirically. In our opinion, this question remains open up to now and is the soft underbelly of Baumgarte's method. Some of the choice of γ can be found, for example, in the papers (Ascher et al., 1995; Chin, 1995).

Further, a deceptive inference can arise that a small denominator in the stabilizing terms causes their large values. But to first order infinitesimal, we have the estimation

$$|\delta \mathbf{Q}| = |\Delta C \mathbf{u}| = \left| \gamma \left(\frac{\partial C}{\partial \mathbf{q}} \cdot \Delta \mathbf{q} \right) \frac{\partial C}{\partial \mathbf{q}} \left(\frac{\partial C}{\partial \mathbf{q}} \right)^{-2} \right| < |\gamma \Delta \mathbf{q}|, \tag{18}$$

where $\Delta \mathbf{q} = \mathbf{q} - \mathbf{q}^{\sim}$ is the difference between an adjacent solution and the reference. In accordance to (18) for a mild γ, the stabilizing terms are of the order of $\Delta \mathbf{q}$ independently of a value of the denominator.

As a particular case of (2), canonical sets have appeal for the stabilization by Hamiltonian, since unlike others these do not imply obtaining the partial derivatives of $C = C(\mathbf{q})$ when stabilized. The fact is that in a stabilized canonical set the partial derivatives are already the known right-hand members of the original set.

Let us write a canonical set in the form

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} + \mathbf{S}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} + \mathbf{T}, \tag{19}$$

where $\mathbf{q}$ and $\mathbf{p}$ are the vectors of generalized coordinates and momenta respectively, $\mathcal{H}$ is the Hamiltonian function, $\mathbf{S}$ and $\mathbf{T}$ are the perturbing terms.

The Hamiltonian function is an integral of the unperturbed part of set (19). Let us use it for stabilizing the perturbed set.

According to the form of set (17), we have the stabilized set

$$\begin{aligned} \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}} + \mathbf{S} - \gamma \Delta \mathcal{H} \frac{\partial \mathcal{H}}{\partial \mathbf{q}} \left[\left(\frac{\partial \mathcal{H}}{\partial \mathbf{q}} \right)^2 + \left(\frac{\partial \mathcal{H}}{\partial \mathbf{p}} \right)^2 \right]^{-1}, \\ \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} + \mathbf{T} - \gamma \Delta \mathcal{H} \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \left[\left(\frac{\partial \mathcal{H}}{\partial \mathbf{q}} \right)^2 + \left(\frac{\partial \mathcal{H}}{\partial \mathbf{p}} \right)^2 \right]^{-1}, \end{aligned} \tag{20}$$

$$\frac{\mathrm{d}\mathcal{H}^{\sim}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \mathbf{q}} \cdot \mathbf{S} + \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \cdot \mathbf{T}, \quad \Delta \mathcal{H} = \mathcal{H}(\mathbf{q}, \mathbf{p}) - \mathcal{H}^{\sim}.$$

If the canonical variables $\mathbf{q}$ and $\mathbf{p}$ are of different dimensions then their subsets have to be stabilized independently. In this case, the stabilized set is

$$\begin{aligned}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial\mathbf{p}} + \mathbf{S} - \gamma_{\mathbf{q}}\Delta\mathcal{H}\frac{\partial\mathcal{H}}{\partial\mathbf{q}}\left(\frac{\partial\mathcal{H}}{\partial\mathbf{q}}\right)^{-2}, \\ \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial\mathbf{q}} + \mathbf{T} - \gamma_{\mathbf{p}}\Delta\mathcal{H}\frac{\partial\mathcal{H}}{\partial\mathbf{p}}\left(\frac{\partial\mathcal{H}}{\partial\mathbf{p}}\right)^{-2}, \end{aligned} \tag{21}$$

$$\frac{\mathrm{d}\mathcal{H}^{\sim}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\mathbf{q}}\cdot\mathbf{S} + \frac{\partial\mathcal{H}}{\partial\mathbf{p}}\cdot\mathbf{T}, \quad \Delta\mathcal{H} = \mathcal{H}(\mathbf{q},\mathbf{p}) - \mathcal{H}^{\sim},$$

where $\gamma_{\mathbf{q}}$ and $\gamma_{\mathbf{p}}$ can be taken independently and, in practice, one of the parameters is not excepted to be zero.

4. Perturbed Two-Body Problem

If the Keplerian energy is considered as the Hamiltonian

$$\mathcal{H} \equiv H_K = \frac{\dot{\mathbf{x}}^2}{2} - \frac{\mu}{r}, \tag{22}$$

then the equations of Keplerian motion can be presented in the canonical form

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \dot{\mathbf{x}} = \frac{\partial H_K}{\partial\dot{\mathbf{x}}}, \\ \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} &= -\frac{\mu}{r^3}\mathbf{x} = -\frac{\partial H_K}{\partial\mathbf{x}}, \end{aligned} \tag{23}$$

where μ is the gravitational parameter and $r = |\mathbf{x}|$ is the radius-vector.

Since we are interested in the stabilization by the energy which is now Hamiltonian at the same time, the canonical notation come in handy for us and we can use the available formulae (21).

Supposing that $\mathbf{q} = \mathbf{x}$, $\mathbf{p} = \dot{\mathbf{x}}$, and $\mathbf{S} = \mathbf{T} = \mathbf{0}$, we obtain the stabilized set

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \dot{\mathbf{x}} + \gamma_{\mathbf{x}}\Delta H_K\frac{\ddot{\mathbf{x}}}{\ddot{\mathbf{x}}^2}, \\ \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} &= \ddot{\mathbf{x}} - \gamma_{\dot{\mathbf{x}}}\Delta H_K\frac{\dot{\mathbf{x}}}{\dot{\mathbf{x}}^2}, \end{aligned} \tag{24}$$

where $\Delta H_K = H_K(\mathbf{x},\dot{\mathbf{x}}) - H_K^{\sim}$ is the deviation of the energy calculated by (22) from the reference $H_K^{\sim} = \mathrm{const}$, and

$$\ddot{\mathbf{x}} \equiv -\frac{\mu}{r^3}\mathbf{x}.$$

In a perturbed problem, the equations have the form

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \dot{\mathbf{x}} = \frac{\partial\mathcal{H}}{\partial\dot{\mathbf{x}}},\\ \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} &= -\frac{\mu}{r^3}\mathbf{x} - \frac{\partial U}{\partial\mathbf{x}} + \mathbf{P} = -\frac{\partial\mathcal{H}}{\partial\mathbf{x}} + \mathbf{T},\end{aligned} \tag{25}$$

where $U = U(\mathbf{x})$ is the potential of conservative perturbing accelerations, $\mathbf{P}$ are nonconservative accelerations, and

$$\mathcal{H} \equiv H = \frac{\dot{\mathbf{x}}^2}{2} - \frac{\mu}{r} + U \tag{26}$$

is the so-called total energy which is an integral of the conservative problem.

Applying (21) again, supposing $\mathbf{S} = \mathbf{0}$ and $\mathbf{T} = \mathbf{P}$, we obtain the stabilized set of the perturbed problem

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \dot{\mathbf{x}} + \gamma_{\mathbf{x}}\Delta H\frac{\ddot{\mathbf{x}}_U}{\ddot{\mathbf{x}}_U^2},\\ \frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} &= \ddot{\mathbf{x}}_U + \mathbf{P} - \gamma_{\dot{\mathbf{x}}}\Delta H\frac{\dot{\mathbf{x}}}{\dot{\mathbf{x}}^2},\\ \frac{\mathrm{d}H^{\sim}}{\mathrm{d}t} &= \dot{\mathbf{x}}\cdot\mathbf{P},\\ \Delta H &= \frac{\dot{\mathbf{x}}^2}{2} - \frac{\mu}{r} + U - H^{\sim},\end{aligned} \tag{27}$$

where

$$\ddot{\mathbf{x}}_U \equiv -\frac{\mu}{r^3}\mathbf{x} - \frac{\partial U}{\partial\mathbf{x}}$$

are the conservative accelerations.

Certainly, the statement on Lyapunov's stability of set (27) is most questionable. It must be proved. But even if the set after stabilization does not get stable then at least one can presume that under weak perturbations the remaining instability should be small enough in comparison with the removed instability. It is clear that the weaker the nonconservative perturbations influence on orbital motion, the higher the efficiency of stabilization.

Concerning numerical aspects in the application of stabilized equations, it should be noted the following. Stabilizing (perturbing) terms are included in the equations of motion. But these are not considered in problem physical definition. In this connection, there arises the question on the correspondence between set (27) and the orbital motion of the problem described by set (25).

From the computational point of view, none of the sets describes the motion adequately since their numerical integration is accompanied by numerical errors. Therefore, here it is appropriate to talk about the degree of the inadequacy when comparing numerical solutions of the sets with the exact.

As practice shows, the stabilized set is integrated with stable (non-increasing) errors in the energy and weakly increasing errors in the coordinates. While the errors of integrating the unstable set grow linearly in the energy and quadratically (superlinearly) in the coordinates.

Because the stabilization is fulfilled by the energy, it guarantees the stability of those orbital elements which are directly bound to the energy. Those are the semi-major axes, the mean motion, or the orbital period. The stability of other elements is doubtful. But, as shown above, this is less important than the energy stability especially if orbital motion are integrated over a long-term interval.

Employing the stabilized set keeps efficient even when the reference energy $H^{\sim}$ starts from an inaccurate value of the initial true energy, i.e. $H_0^{\sim} \approx H(\mathbf{x}_0, \dot{\mathbf{x}}_0)$. Estimations (1) show that the constant error ΔH mainly results in the linear growth of the errors $\Delta\mathbf{x}$ and therefore a minor variance between the initial energies is unable to change substantially the behaviour of the numerical error $\Delta\mathbf{x}$. Nevertheless, the equality remains preferable. In light of this specificity of stabilized set (27), the reference energy $H^{\sim}$ can be considered as a variable directly independent on the coordinates and their derivatives and as a parameter in the unperturbed case.

In this sense the well-known equations in the Keplerian elements are stabilized as well because the integrated variables bound with the energy are considered here as independent ones. In unperturbed motion, these turn into constant parameters. Then the whole set of the equations get stable by all the integrated variables excepting the energy ones. One can say the same, for example, about the sets of Kustaanheimo-Stiefel theory (Stiefel, Scheifele, 1971) or the sets of Sperling-Burdet equations (Burdet, 1968).

5. Almost Circular Motion

A simple method for stabilizing the equations of almost circular motion can be proposed if the radius-vector r is considered as an independent variable.

In the circular unperturbed motion, $r = \text{const}$ and the equations of motion have the form of the unperturbed harmonic oscillator

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} + n'^2\mathbf{x} = 0, \quad n' \equiv \sqrt{\frac{\mu}{r^3}} = n = \text{const}, \tag{28}$$

which, as is known, is stable by Lyapunov.

In weakly perturbed almost circular motion $n' \neq$ const and the equations of motion become weakly nonlinear:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} + n'^2\mathbf{x} = -\frac{\partial U}{\partial \mathbf{x}} + \mathbf{P}. \tag{29}$$

To keep the independence status of r, the set has to be supplemented with the r-equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r}\mathbf{x}\cdot\dot{\mathbf{x}}. \tag{30}$$

In general, the set of equations (29) and (30) is unstable. However, under small eccentricities the remaining instability after stabilization should be small enough in comparison with the eliminated instability.

6. Three-Body Problem

The three-body problem equations can be written in a canonical form as well and stabilized by using formulae (21).

Let us consider the motion of three bodies with the masses m_1, m_2 and m_3 in their gravitational field with respect to a barycentric coordinate frame. Then the equations of the dynamical system are

$$\begin{aligned}
m_1\frac{\mathrm{d}^2\mathbf{x}_1}{\mathrm{d}t^2} &= -k^2 m_1 m_2\frac{\mathbf{x}_1-\mathbf{x}_2}{\Delta_{12}^3} - k^2 m_1 m_3\frac{\mathbf{x}_1-\mathbf{x}_3}{\Delta_{13}^3},\\
m_2\frac{\mathrm{d}^2\mathbf{x}_2}{\mathrm{d}t^2} &= -k^2 m_2 m_3\frac{\mathbf{x}_2-\mathbf{x}_3}{\Delta_{23}^3} - k^2 m_2 m_1\frac{\mathbf{x}_2-\mathbf{x}_1}{\Delta_{21}^3},\\
m_3\frac{\mathrm{d}^2\mathbf{x}_3}{\mathrm{d}t^2} &= -k^2 m_3 m_1\frac{\mathbf{x}_3-\mathbf{x}_1}{\Delta_{31}^3} - k^2 m_3 m_2\frac{\mathbf{x}_3-\mathbf{x}_2}{\Delta_{32}^3},
\end{aligned} \tag{31}$$

where $\Delta_{ij} = |\mathbf{x}_i - \mathbf{x}_j| \quad (i,j = 1,3)$, and k^2 is the Gaussian universal constant.

If we go over to first order equations and introduce the 9-dimensional vectors $\mathbf{q} \equiv (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)^T$ and $\mathbf{p} \equiv (m_1\dot{\mathbf{x}}_1, m_2\dot{\mathbf{x}}_2, m_3\dot{\mathbf{x}}_3)^T$ then we can write set (31) in the canonical form

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\mathbf{p}}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\mathbf{q}}, \tag{32}$$

with the Hamiltonian

$$\mathcal{H} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} - k^2\frac{m_1m_2}{\Delta_{12}} - k^2\frac{m_2m_3}{\Delta_{23}} - k^2\frac{m_3m_1}{\Delta_{31}}. \tag{33}$$

Let us apply Baumgarte's stabilization by Hamiltonian to (32). The condition $\gamma_{\mathbf{q}} = 0$ allows us to come back to the set of second order equations. As a result, the stabilized set takes the form

$$\begin{aligned}
\frac{\mathrm{d}^2\mathbf{x}_1}{\mathrm{d}t^2} &= -k^2 m_1 m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{\Delta_{12}^3} - k^2 m_1 m_3 \frac{\mathbf{x}_1 - \mathbf{x}_3}{\Delta_{13}^3} - \gamma \Delta\mathcal{H} \frac{\dot{\mathbf{x}}_1}{V^2}, \\
\frac{\mathrm{d}^2\mathbf{x}_2}{\mathrm{d}t^2} &= -k^2 m_2 m_3 \frac{\mathbf{x}_2 - \mathbf{x}_3}{\Delta_{23}^3} - k^2 m_2 m_1 \frac{\mathbf{x}_2 - \mathbf{x}_1}{\Delta_{21}^3} - \gamma \Delta\mathcal{H} \frac{\dot{\mathbf{x}}_2}{V^2}, \\
\frac{\mathrm{d}^2\mathbf{x}_3}{\mathrm{d}t^2} &= -k^2 m_3 m_1 \frac{\mathbf{x}_3 - \mathbf{x}_1}{\Delta_{31}^3} - k^2 m_3 m_2 \frac{\mathbf{x}_3 - \mathbf{x}_2}{\Delta_{32}^3} - \gamma \Delta\mathcal{H} \frac{\dot{\mathbf{x}}_3}{V^2},
\end{aligned} \tag{34}$$

where $V^2 = \dot{\mathbf{x}}_1^2 + \dot{\mathbf{x}}_2^2 + \dot{\mathbf{x}}_3^2$, and $\Delta\mathcal{H}$ is the deviation of the integral function determined by (33) from the reference $\mathcal{H}^\sim = \text{const}$.

Formally, we have stabilized the set of the three-body problem equations which now are asymptotically stable by the Hamiltonian of the problem. However, the question on the efficiency of the stabilization as a remedy for Lyapunov's instability remains open. So, for example, if the mass m_3 is much less than two others, Hamiltonian (33) gets less sensitive to integration errors in the orbit of third body. In that case, stabilization would rather work only for the equations of two massive bodies and what is the worst thing is that through the stabilizing terms the errors of the two-body dynamical system would partly spill over into the numerical orbit of third body.

7. Integrals Against Lyapunov's Instability

It is clear at least intuitively that, in general, the asymptotic stability by an integral (parametric) variable is not sufficient condition for Lyapunov's stability. We show it for the case of a simple canonical set with Hamiltonian $\mathcal{H} = \mathcal{H}(\mathbf{q})$

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\mathbf{p}} = \mathbf{0}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\mathbf{q}} \equiv \boldsymbol{\omega}(\mathbf{q}), \tag{35}$$

where we will call $\boldsymbol{\omega}$ frequencies.

Set (35) is integrated analytically and has the simple solution

$$\mathbf{q} = \mathbf{q}_0 = \mathbf{const}, \quad \mathbf{p} = \boldsymbol{\omega}(\mathbf{q})(t - t_0) + \mathbf{p}_0, \tag{36}$$

where $\mathbf{q}_0 = \mathbf{q}(t_0)$ and $\mathbf{p}_0 = \mathbf{p}(t_0)$.

Let us take some reference solution $(\mathbf{q}^\sim, \mathbf{p}^\sim)$:

$$\mathbf{q}^\sim = \mathbf{q}_0^\sim = \mathbf{const}, \quad \mathbf{p}^\sim = \boldsymbol{\omega}(\mathbf{q}^\sim)(t - t_0) + \mathbf{p}_0^\sim, \tag{37}$$

and estimate the behaviour of any adjacent solution $(\mathbf{q}, \mathbf{p})$ relative to the reference by the differences $\Delta\mathbf{q} = \mathbf{q} - \mathbf{q}^{\sim}$ and $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}^{\sim}$ caused by the initial differences $\Delta\mathbf{q}_0 = \mathbf{q}_0 - \mathbf{q}_0^{\sim}$ and $\Delta\mathbf{p}_0 = \mathbf{p}_0 - \mathbf{p}_0^{\sim}$.

Subtracting (37) from (36) gives

$$\Delta\mathbf{q} = \Delta\mathbf{q}_0 = \mathbf{const}, \quad \Delta\mathbf{p} = \Delta\boldsymbol{\omega}\Delta t + \Delta\mathbf{p}_0, \tag{38}$$

where $\Delta\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{q}) - \boldsymbol{\omega}(\mathbf{q}^{\sim})$ and $\Delta t = t - t_0$.

Generally speaking, $\Delta\boldsymbol{\omega} \neq 0$ and the difference $\Delta\mathbf{p}$ must increase with time unboundedly. Consequently, the set is unstable by Lyapunov. Evidently, for stabilization it is necessary to demand $\Delta\boldsymbol{\omega} \to 0$ so that, as a result, we could have

$$\boldsymbol{\omega}(\mathbf{q}) = \boldsymbol{\omega}(\mathbf{q}^{\sim})$$

or

$$\left.\frac{\partial\mathcal{H}}{\partial\mathbf{q}}\right|_{\mathbf{q}} = \left.\frac{\partial\mathcal{H}}{\partial\mathbf{q}}\right|_{\mathbf{q}^{\sim}}. \tag{39}$$

But suppose that we stabilize the set by Hamiltonian and demand from the adjacent solution to satisfy the relation

$$\mathcal{H}(\mathbf{q}) = \mathcal{H}(\mathbf{q}^{\sim}). \tag{40}$$

However, (40) does not cause (39). One can see it in a special case.

Let us consider the 2-frequency canonical set with the Hamiltonian

$$\mathcal{H} = \mathcal{H}(q_1, q_2) = q_1^2 + q_2^2.$$

Suppose the generalized coordinates of $\mathbf{q}$ and $\mathbf{q}^{\sim}$ have the values

$$\begin{aligned} q_1 &= 0, \quad q_2 = \varepsilon, \\ q_1^{\sim} &= \varepsilon, \quad q_2^{\sim} = 0, \end{aligned} \tag{41}$$

where ε is a small constant. It is not difficult to see that solutions (41) satisfy (40) but these do not do (39).

It should be noted, (39) results from (40) only for one-frequency canonical sets including the Keplerian.

In the Keplerian case, by using the known analytical solutions, we succeeded in obtaining the estimations which pointed out to what kind of integrals we need to use for the stabilization as a remedy for Lyapunov's instability. These are the integral of the Keplerian energy and those which are bound up with the energy. Stabilization by other integrals is less efficient as it does not eliminate quick (quadratic) drift of numerical errors along orbit.

Stabilization in other celestial mechanics problems is employed actually blindly because its efficiency is not predictable analytically and may be tested only numerically from multifold experiments. In this connection, a skeptical attitude to the stabilization in the three-body problem is quite reasonable.

8. Perturbed Restricted Three-Body Problem

Now we consider a particular case of the three-body problem in which the third body is assumed to have the neglible mass m_3. This problem is of practical interest since its stabilized equations can be useful for simulating, for example, the dynamics of asteroids approaching to the major planets.

The obstacle in stabilization way here consists in that all the available integrals of the general three-body problem degenerate into those of the two-body problem and, obviously, therefore to stabilize the equations of the restricted three-body problem by these integrals is of no use at all. Although, there appears an integral in the particular case of the restricted problem when two massive bodies move on circular orbits and the motion of the third body is considered in the rotating frame. This is Jacobi integral. We make use of this for stabilizing the equations of the restricted problem.

We consider the motion of a celestial body with neglible mass in the gravitational field of two massive bodies with the masses m_1 and m_2 under the action of the perturbing force $\mathbf{P}$. Then, in a stationary frame, the differential equations of motion take the form

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\frac{\mu_1}{\Delta_1^3}(\mathbf{x}-\mathbf{x}_1) - \frac{\mu_2}{\Delta_2^3}(\mathbf{x}-\mathbf{x}_2) + \mathbf{P}, \tag{42}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}$ are the position vectors of the first, the second, and the third bodies respectively, $\Delta_1 = |\mathbf{x}-\mathbf{x}_1|$ and $\Delta_2 = |\mathbf{x}-\mathbf{x}_2|$ are the distances of the third body to the massive ones, and $\mu_1 = k^2 m_1$ and $\mu_2 = k^2 m_2$ are the gravitational parameters.

If one introduce the potential function

$$W = \frac{\mu_1}{\Delta_1} + \frac{\mu_2}{\Delta_2}, \tag{43}$$

then equations (42) can be presented as

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{\partial W}{\partial \mathbf{x}} + \mathbf{P}. \tag{44}$$

Now we go over to the frame rotating with a constant frequency ω about the applicate axis.

The rotation transformation by an angle φ can be written in the form

$$\mathbf{x}_R = \mathbf{R}(\varphi)\mathbf{x}, \quad \text{where} \quad \mathbf{R}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{45}$$

$\mathbf{x}_R$ and $\mathbf{x}$ is the position vectors in the rotating and stationary frames respectively. Hence, the relations for velocities and accelerations are

$$\dot{\mathbf{x}}_R = \omega\mathbf{I}\mathbf{x}_R + \mathbf{R}(\varphi)\dot{\mathbf{x}}, \tag{46}$$

$$\ddot{\mathbf{x}}_R = 2\omega\mathbf{I}\dot{\mathbf{x}}_R - \omega^2\mathbf{I}^2\mathbf{x}_R + \mathbf{R}(\varphi)\ddot{\mathbf{x}}, \tag{47}$$

where $\omega = \dot{\varphi} = \mathrm{const}$, and

$$\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{moreover,} \quad \mathbf{RI} = \mathbf{IR} = \mathbf{R}'(\varphi).$$

Since

$$\frac{\partial W}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}_R}{\partial \mathbf{x}}\frac{\partial W}{\partial \mathbf{x}_R} = \mathbf{R}^T(\varphi)\frac{\partial W}{\partial \mathbf{x}_R},$$

it follows from (44) that

$$\ddot{\mathbf{x}} = \mathbf{R}^T(\varphi)\frac{\partial W}{\partial \mathbf{x}_R} + \mathbf{P}. \tag{48}$$

Let us substitute (48) into (47) and abandon the index R. Employing the property of orthonormal matrices according to which $\mathbf{R}^T(\varphi) = \mathbf{R}^{-1}(\varphi)$, we obtain the differential equations of the problem in the rotating frame

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = 2\omega\mathbf{I}\dot{\mathbf{x}} - \omega^2\mathbf{I}^2\mathbf{x} + \frac{\partial W}{\partial \mathbf{x}} + \mathbf{R}(\varphi)\mathbf{P}. \tag{49}$$

Set (49) yields Jacobi equation

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\dot{\mathbf{x}}_1 \cdot \frac{\partial W}{\partial \mathbf{x}_1} - \dot{\mathbf{x}}_2 \cdot \frac{\partial W}{\partial \mathbf{x}_2} + \dot{\mathbf{x}} \cdot \mathbf{RP}, \tag{50}$$

where

$$\Omega = \frac{\dot{\mathbf{x}}^2}{2} + \frac{\omega^2}{2}\mathbf{x} \cdot \mathbf{I}^2\mathbf{x} - W. \tag{51}$$

In the unperturbed case, if the massive bodies are immovable in the rotating frame:

$$\dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_2 = 0, \tag{52}$$

equation (50) gives Jacobi integral

$$\Omega = \mathrm{const.} \tag{53}$$

The condition of immovability (52) occurs in the circular problem when the massive bodies move relative to each other on circular Keplerian orbits with the mean motions $n = \omega$ in the orbital plane perpendicular to the applicate axis.

As stated above, we make use of Jacobi integral for stabilizing the equations of motion (49). At first, we rewrite the equations in the form of the set of first order equations and then, as in the canonical cases, apply Baumgarte's method to the equations of the coordinates $\mathbf{x}$ and its velocities $\dot{\mathbf{x}}$ independently. As a result, the stabilized set takes the form

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \dot{\mathbf{x}} - \gamma_{\mathbf{x}}\Delta\Omega\frac{\ddot{\mathbf{x}}_\omega}{\ddot{\mathbf{x}}_\omega^2}, \\
\frac{\mathrm{d}\dot{\mathbf{x}}}{\mathrm{d}t} &= 2\omega\mathbf{I}\dot{\mathbf{x}} - \ddot{\mathbf{x}}_\omega + \mathbf{RP} - \gamma_{\dot{\mathbf{x}}}\Delta\Omega\frac{\dot{\mathbf{x}}}{\dot{\mathbf{x}}^2}, \\
\frac{\mathrm{d}\Omega^\sim}{\mathrm{d}t} &= -\dot{\mathbf{x}}_1\cdot\frac{\partial W}{\partial \mathbf{x}_1} - \dot{\mathbf{x}}_2\cdot\frac{\partial W}{\partial \mathbf{x}_2} + \dot{\mathbf{x}}\cdot\mathbf{RP}, \\
\Delta\Omega &= \frac{\dot{\mathbf{x}}^2}{2} + \frac{\omega^2}{2}\mathbf{x}\cdot\mathbf{I}^2\mathbf{x} - W - \Omega^\sim,
\end{aligned} \tag{54}$$

where

$$\ddot{\mathbf{x}}_\omega \equiv \omega^2\mathbf{I}^2\mathbf{x} - \frac{\partial W}{\partial \mathbf{x}}.$$

It should be taken into consideration a useful relation

$$\frac{\partial W}{\partial \mathbf{x}_1} + \frac{\partial W}{\partial \mathbf{x}_2} = \frac{\mu_1}{\Delta_1^3}(\mathbf{x}-\mathbf{x}_1) + \frac{\mu_2}{\Delta_2^3}(\mathbf{x}-\mathbf{x}_2) = -\frac{\partial W}{\partial \mathbf{x}}.$$

It is evident that the efficiency of stabilization should be high when the perturbing force $\mathbf{P}$ and the velocities $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ are small. Because most major planets move on almost circular orbits, these conditions can occur in asteroid and comet problems where the motion of a celestial body under consideration submits mainly to the gravitational influence from the Sun and some major planet while the influence from the other planets is slight and included into perturbing forces. In the problems, the motion is considered in a rotating ecliptic frame with a constant frequency close to the mean motion of the second planet.

If obtaining the velocities $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ is problematically then one can propose another way for taking into account the movability of the massive bodies.

We rewrite the gravitational accelerations of the unperturbed problem as

$$\frac{\partial W}{\partial \mathbf{x}} = \frac{\partial W_F}{\partial \mathbf{x}} + \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial W_F}{\partial \mathbf{x}} = \frac{\partial W_F}{\partial \mathbf{x}} + \mathbf{P}_F,$$

where

$$\mathbf{P}_F = \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial W_F}{\partial \mathbf{x}},$$

and W_F is the potential (43) in which the coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$ are fixed. Hence, the movability effect can be considered as a perturbing acceleration. Introducing it into the problem yields a new differential equation of Jacobi function

$$\frac{\mathrm{d}\Omega_F^{\sim}}{\mathrm{d}t} = \dot{\mathbf{x}} \cdot (\mathbf{R}\mathbf{P} + \mathbf{P}_F), \tag{55}$$

where

$$\Omega_F^{\sim} = \frac{\dot{\mathbf{x}}^2}{2} + \frac{\omega^2}{2}\mathbf{x} \cdot \mathbf{I}^2\mathbf{x} - W_F.$$

When using equations (54), it is necessary to replace $\Omega^{\sim}$ by $\Omega_F^{\sim}$ and W by W_F in the stabilizing terms.

In the end, it is interesting to note that, as is known, Jacobi integral exists not only in the problem we have considered just now but also in any other one where gravitational field is stationary in rotating frame and, therefore, the stabilized set (54) can be used. For instance, it is quite suitable for describing near satellite dynamics where the rotating frame is concerned with planet rotation. In fact, suppose that W in (54) represents as the gravitational field of the central planet. If the planet rotates evenly around an immovable axis then, in the rotating frame with the planet rotation frequency, the potential W is the function of only the position vector and, therefore, there exists an analogue to integral (53) which can be employed for stabilization. If the planet rotation is weakly uneven then the unevenness effect should be considered as a perturbing acceleration like above.

9. Numerical Results

Stabilization is an excellent means for improving the efficiency of numerical integration. Employing stabilization methods in celestial mechanics problems allows one to raise the accuracy of integrating orbits by several orders and higher. Various experiments evidence it (Nacozy, 1971; Baumgarte, 1973; Murison, 1989).

Here we show the ability of stabilization applied to simulating the orbital dynamics of concrete celestial objects: Amalthea, Thebe, Io, Europa, Ganymede and Callisto (near Jovian satellites)[1]; Alinda (asteroid) and the major planets of the Solar system. We also resort to hypothetical objects: an outer planet in a binary star and a three-star dynamical system.

Simulations of their orbits have been realized in FORTRAN-code and based on equations (27), (34) and (54) integrated numerically by 15-order Everhart's method (Everhart, 1974) with variable step size in double-precision arithmetic. The step size in the integration method is selected by a given parameter L so that the last term in the approximating expansion has magnitude 10^{-L} (Everhart, 1974). Therefore, the larger the parameter, the less the step size.

The cardinal gravitational forces are taken into account in the simulations: from the gravitational field of Jupiter up to the second zonal harmonics, Galilean satellites and the Sun for the near Jovian satellites; from the Sun and the major planets for Alinda; from the gravitational field of only two stars for their outer planet and mutual attractions for both the three-star system and the planetary system. In the satellite simulations, coordinates of Galilean satellites are obtained from a simplified Lieske's theory (Rohde, Sinclair, 1992) and those of the Sun are calculated by the formulae of circular motion. In the asteroid simulation, coordinates of both the Sun and the major planets are obtained from DE200/LE200.

We have simulated all the orbits over certain time intervals in stabilized ($\gamma \neq 0$) and non-stabilized ($\gamma = 0$) cases and then analyzed their accuracy and simulating speed.

The accuracy (Δr) of simulating here is evaluated in position vector by the well-known forward-and-backward integration method. The number of integrating steps (NS) over a whole interval of integration is taken as an indicator of simulating speed.

We have investigated stabilization in detail applied to Amalthea (JV: $T = 0.5$ days, $e = 0.003$, $i = 0.3°$).

In fig. 1. the evolution of numerical errors is given for integrating the stabilized and non-stabilized equations of motion (27) at $L = 10$ over a time interval 1000 years (500 forward and 500 backward). As is evident from the figure, the unstable error engirds twice the orbit. From the beginning, when the error is small, it grows quadratically. According to (1), such a systematic behaviour of the error is explained by a linear growth of the error in the energy ΔH (Fig. 2.). Afterwards, the error Δr spreads along the orbit at a raising speed proportionate to ΔH.

[1] Initial data for the satellites have been taken from (Jacobson, 1994; Lieske, 1998)

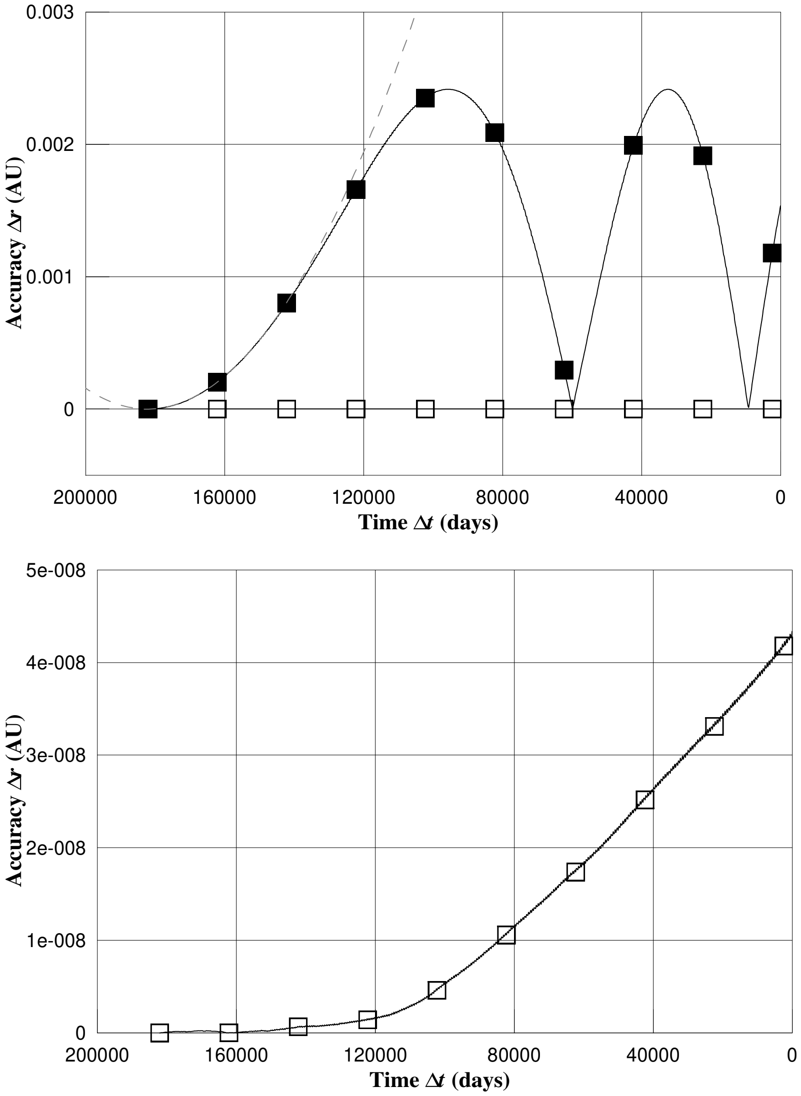

Figure 1. Evolution of integration errors in position vector: stabilized — □, non-stabilized — ■. Amalthea: 1000 years. The under plot is for the stabilized error in vast scale.

While the stabilized error grows much slower than the non-stabilized and keeps small enough. We would like to notice that its systematic behaviour is rather caused by the remaining instability.

In the under plot of fig. 2. one can see how stabilization continuously corrects the orbit for its deviation from the reference integral surface.

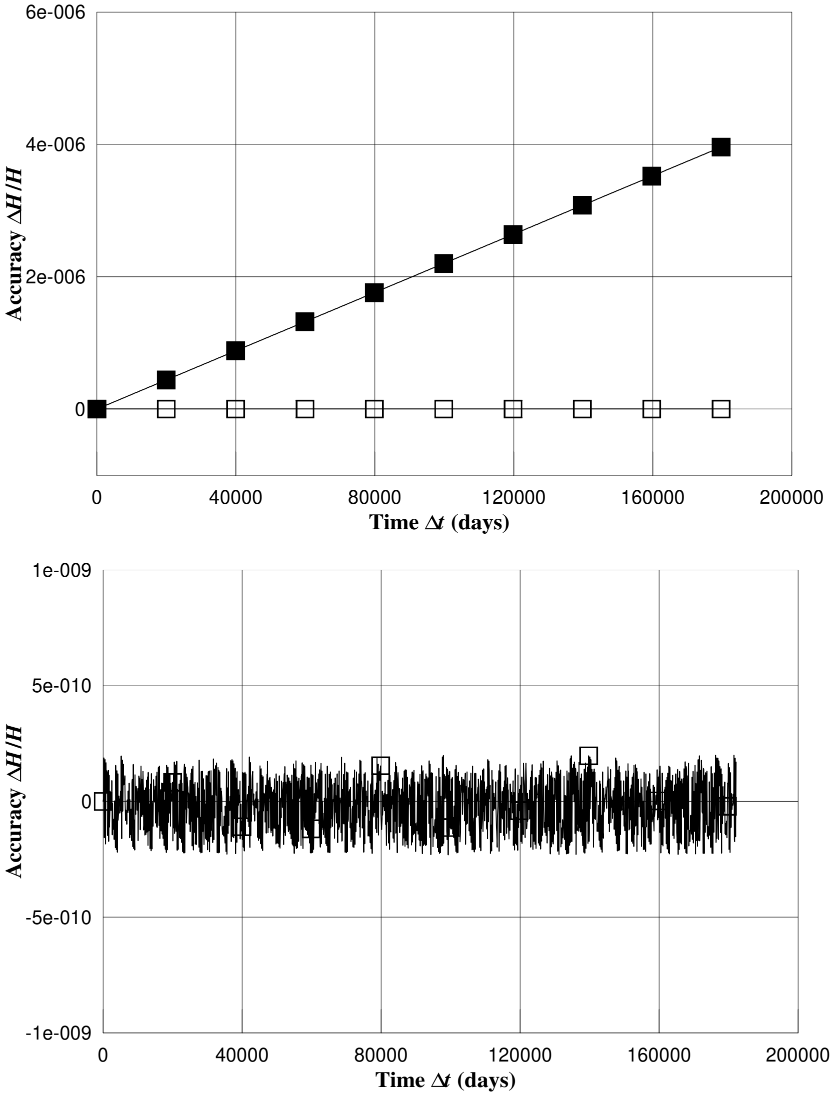

Figure 2. Evolution of integration errors in energy: stabilized — □, non-stabilized — ■. Amalthea: 1000 years. The under plot is for the stabilized error in vast scale.

The numerical results above have been obtained for the optimal stabilizing parameter $\gamma = 13$ at which the stabilized integration error becomes minimum[2]. In order to find it, we have varied the parameter from 0 to 30 and evaluated integration accuracy over a time interval

[2] Hereafter, for stabilized simulations of satellite motion we have taken $\gamma_{\mathbf{x}} = 0$, and $\gamma = \gamma_{\dot{\mathbf{x}}}$.

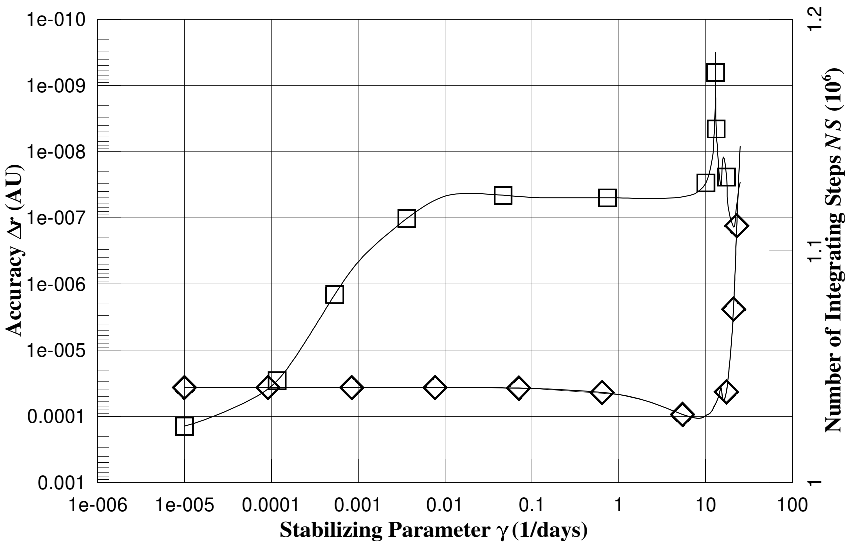

Figure 3. Accuracy (□: left scale) and simulating speed (◇: right scale) v. stabilizing parameter. Amalthea: 100 years.

100 years (50 forward and 50 backward). The accuracy dependence on γ is presented in fig. 3.

The figure shows the accuracy improves as the parameter increases till 13 after that the accuracy falls down abruptly. It should be noted that the integrating speed slightly rises in the vicinity of the optimal γ after which, however, it noticeably gets worse.

Using the optimal γ, we further have varied L to obtain accuracy-speed characteristics for the stabilized and non-stabilized simulations of the satellite motion. They are given in fig. 4.

One can see from these that when the simulating speed is the highest (NS is the smallest) the stabilized simulation exceeds its non-stabilized rival in accuracy by almost 5 orders. Although, the excess decreases as the simulating speed goes down (NS increases). Finally, both characteristics get comparable because the simulations are oversaturated by round-off errors. However, notice that the characteristic for the stabilized simulation remains on a specific level within an order for any NS. From this we guess that stabilization efficiently eliminates only truncation errors but not round-off errors.

The characteristics can be interpreted in a different way. These also show how much stabilization is able to raise the simulating speed: high accuracy is attained by the stabilized simulation approximately 3 times faster than by the non-stabilized.

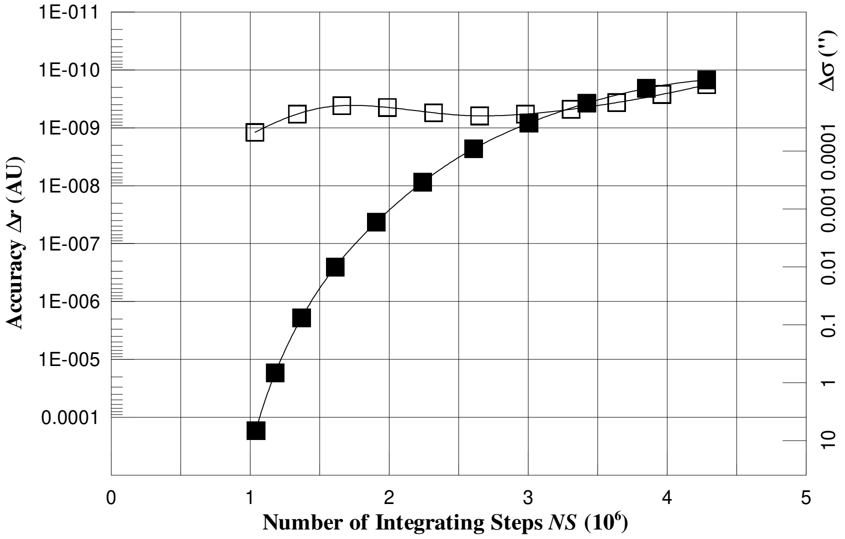

Figure 4. Accuracy-speed characteristics for stabilized (□) and non-stabilized (■) simulations. Amalthea: 100 years. The accuracy in arc relative the Earth is given on the right scale.

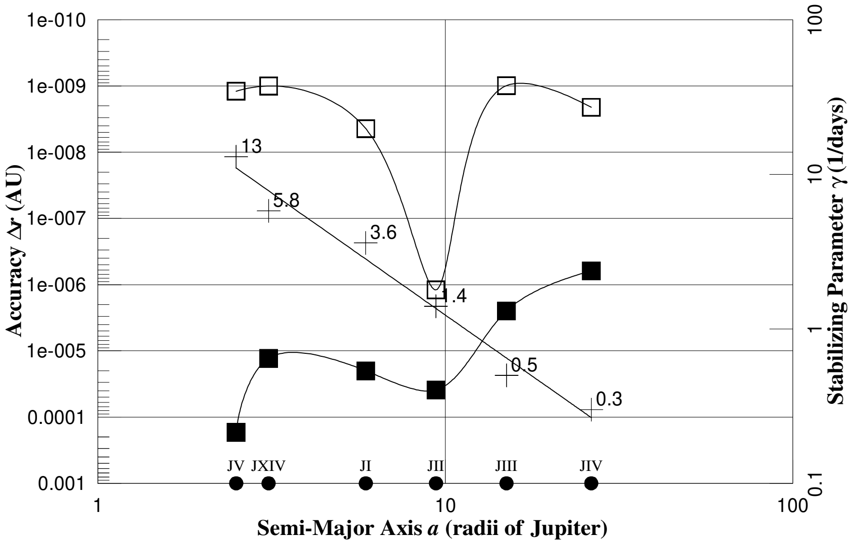

Figure 5. Accuracies of simulating orbits (stabilized — □, non-stabilized — ■: left scale) and optimal stabilizing parameters (+: right scale) v. remoteness of satellites from Jupiter. Near Jovian satellites: 100 years.

Numerical experiments made for the other near satellites have also revealed high efficiency of stabilization. In the fig. 5 the accuracies of simulating the satellite orbits are presented.

The distinguishing feature of the Jovian satellites under consideration consists in that they move on almost circular and equatorial orbits in powerful gravitational field of the central planet under the influence from Galilean satellites.

So we see that the highest efficiency of stabilization is achieved for Amalthea and Thebe since their orbits are weakly perturbed by Galilean satellites despite the fact that both the inner satellites are in powerful but conservative gravitational field of the planet. While the second Galilean satellite, Europa, is strongly perturbed by the three others and, therefore, the efficiency of stabilization is fairly low here. On the whole, the application of stabilization to the satellite orbits (except Europa) is able to raise the accuracy of simulating by 3 orders and higher.

In the figure, there is a plot that presents magnitudes of the optimal stabilizing parameter at which we have obtained the accuracies of the stabilized simulations. From this we find out that the optimal parameter should be approximately equal to the mean motion of satellite orbit.

In the next figure, we present numerical results of an experiment on investigating the stabilization for almost circular orbit (Section 5) which we will call r-stabilization. The object of our experiment has been to evaluate the accuracy and speed of simulating the family of two-body orbits of hypothetical Jovian satellites with various eccentricities $e = 0$–0.01 but with a certain semi-major axis $a = 2.5$ radii of Jupiter (similar to that of Amalthea). The equations of motion have been integrated at $L = 10$.

As is obvious, r-stabilization allows us to obtain the highest accuracy for the circular orbit ($e = 0$). However, as the eccentricity grows, the accuracy of the stabilized simulation falls down swiftly till it gets comparable with that of the non-stabilized simulation, i.e. at $e = 0.003$. Although, the speed of the stabilized simulation remains invariable and exceeds that of the non-stabilized simulation almost by 15% right up to $e = 0.005$. It explains why the accuracy of r-stabilization keeps falling on the interval $e = 0.003$–0.005 (Fig. 6).

Further increase of the eccentricity on the interval $e = 0.005$–0.01 results in that the speed of stabilized simulating goes down and r-stabilization makes up for its once lost efficiency in accuracy. Thus, r-stabilization efficiently works only for the orbits with the eccentricities $e = 0$–0.001. Say, for Amalthea ($e = 0.003$) we can achieve only

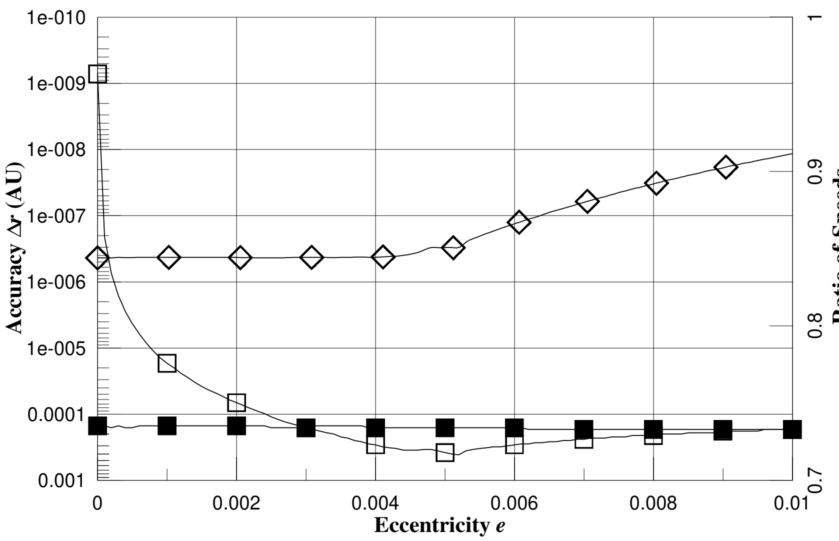

Figure 6. Efficiency of r-stabilization (stabilized — □, non-stabilized — ■: left scale; ratio of simulating speeds ◇: right scale). Near satellites with almost circular orbits: 100 years.

an insignificant increase of simulating speed without any change in accuracy.

Not less interesting results have we obtained for Baumgarte's stabilization by Jacobi integral in the restricted three-body problem applied to Alinda and the outer planet of a hypothetical binary star.

The feature of Alinda's orbit consists in that it undergoes sufficiently strong gravitational influence from Jupiter (but not close encounters with the planet) (Fig. 7.) and, therefore, it can be interpreted by the restricted three-body problem where Jupiter is taken as the second body.

By varying L, we have calculated accuracy-speed characteristics for the stabilized and non-stabilized simulations of Alinda's motion in both stationary and rotating frames over a time interval about 100 years (50 forward and 50 backward). These are given in fig. 8.

What is first of all striking is high accuracy of integration in the rotating frame. It is most likely to be so because of that the orbital motion of the asteroid is integrated in the rotating frame better, from the computational point of view, than in the stationary.

Further, one can see that stabilization raises the accuracy of simulating approximately by an order (in the rotating frame slightly higher). Nevertheless, taking into account the specific character of integration in the rotating frame, we recommend using the three-body stabilization.

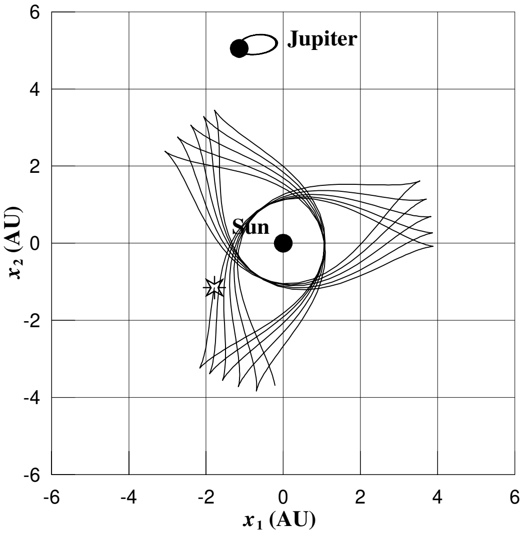

Figure 7. Alinda's orbit in rotating frame ($\Delta t = 60$ years)

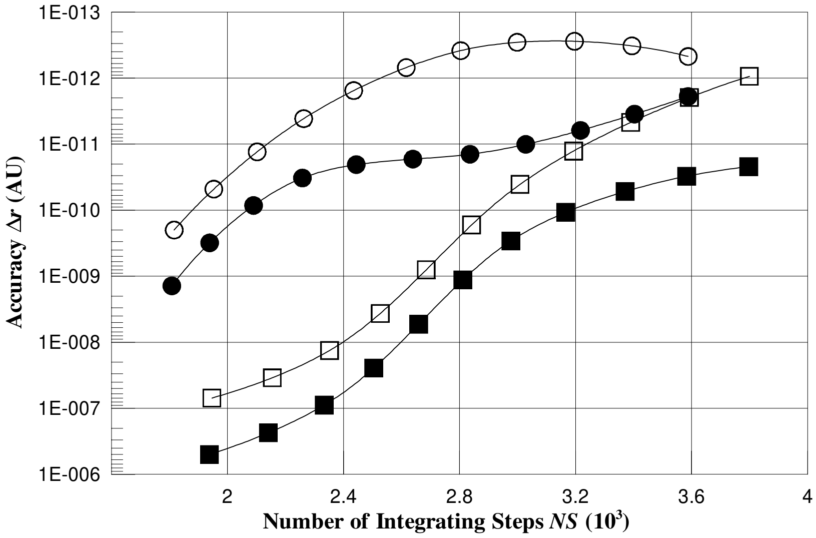

Figure 8. Accuracy-speed characteristics for stabilized (in stationary frame — □, in rotating frame — ○) and non-stabilized (in stationary frame — ■, in rotating frame — ●) simulations. Alinda: ~ 100 years.

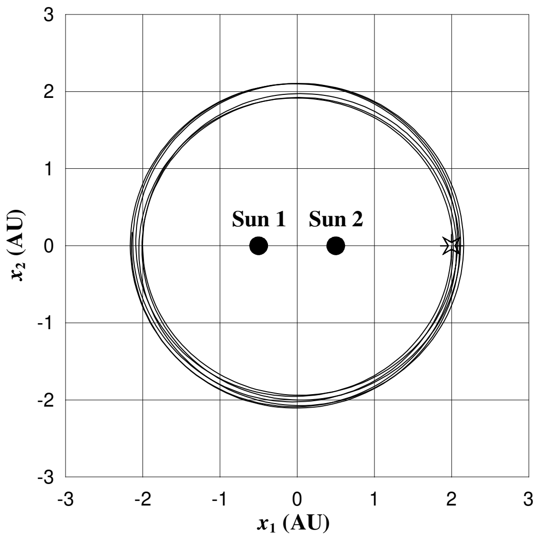

Figure 9. Orbit of planet of binary star in rotating frame ($\Delta t = 7$ years)

Low efficiency of the three-body stabilization is possible to be explained by the noticeable movability of Jupiter in the rotating frame (Fig. 7.). In the next problem, we have fixed both the first and second bodies.

Over a time interval about 14 years (7 forward and 7 backward), we have considered the motion of a planet orbiting in the gravitational field of two stars with solar masses which, in turn, move relative to each other on Keplerian circular orbits ($T = 258.3$ days) at the distance 1 AU (Fig. 9.).

Here the efficiency of the three-body stabilization has been appear to be more impressive than above (Fig. 10.). This allows us to raise the accuracy of integration by 2–3 orders while the two-body stabilization does not higher than by an order. On the whole, the characteristic for the three-body stabilization is able to exceed the others in accuracy almost by 3 orders.

Further we examine stabilization in the general three-body problem. Let us assume that in some three-star system two stars have the masses $m_1 = m_2 = 1$ (in solar masses) and consider three cases when the mass of thrid star possesses the values $m_3 = 0.2, 0.1, 0.02$. For simulating their orbits (Fig. 11.) equations (34) stabilized by energy (33) have been taken. We have integrated these ($L = 8$) and evaluated integration errors over time interval 1000 years (500 forward and 500 backward) at various stabilizing parameters γ. The errors are presented in fig. 12.

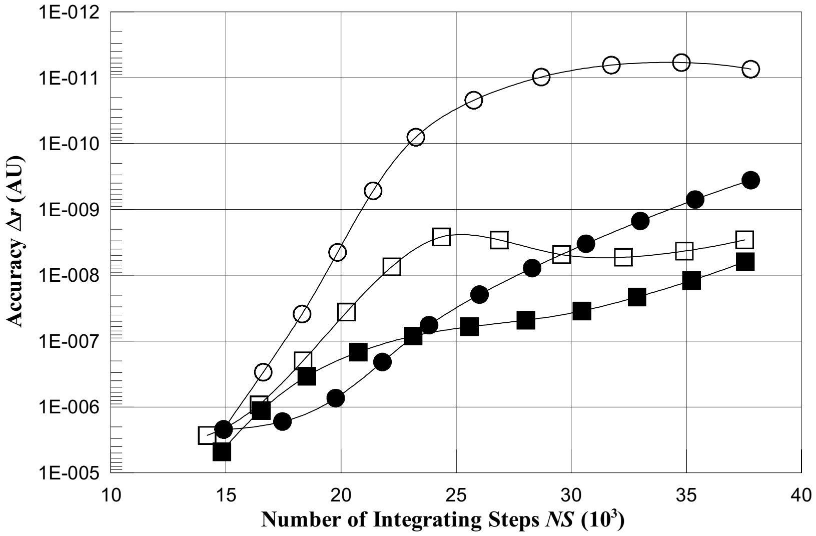

Figure 10. Accuracy-speed characteristics for stabilized (in stationary frame — □, in rotating frame — ○) and non-stabilized (in stationary frame — ■, in rotating frame — ●) simulations. Planet of binary star: ∼ 14 years.

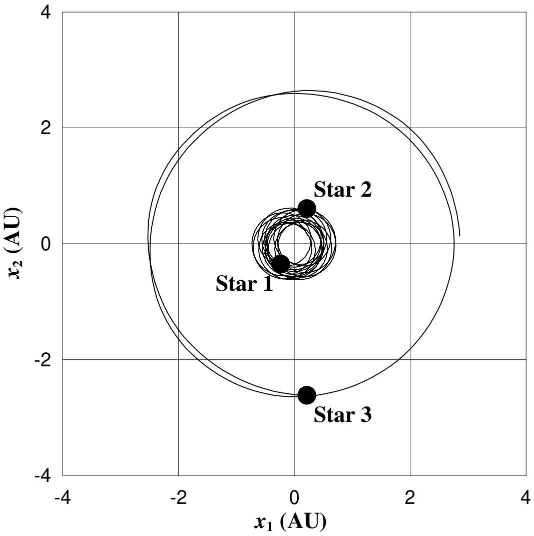

Figure 11. Orbits of three-star system ($\Delta t = 5.5$ years)

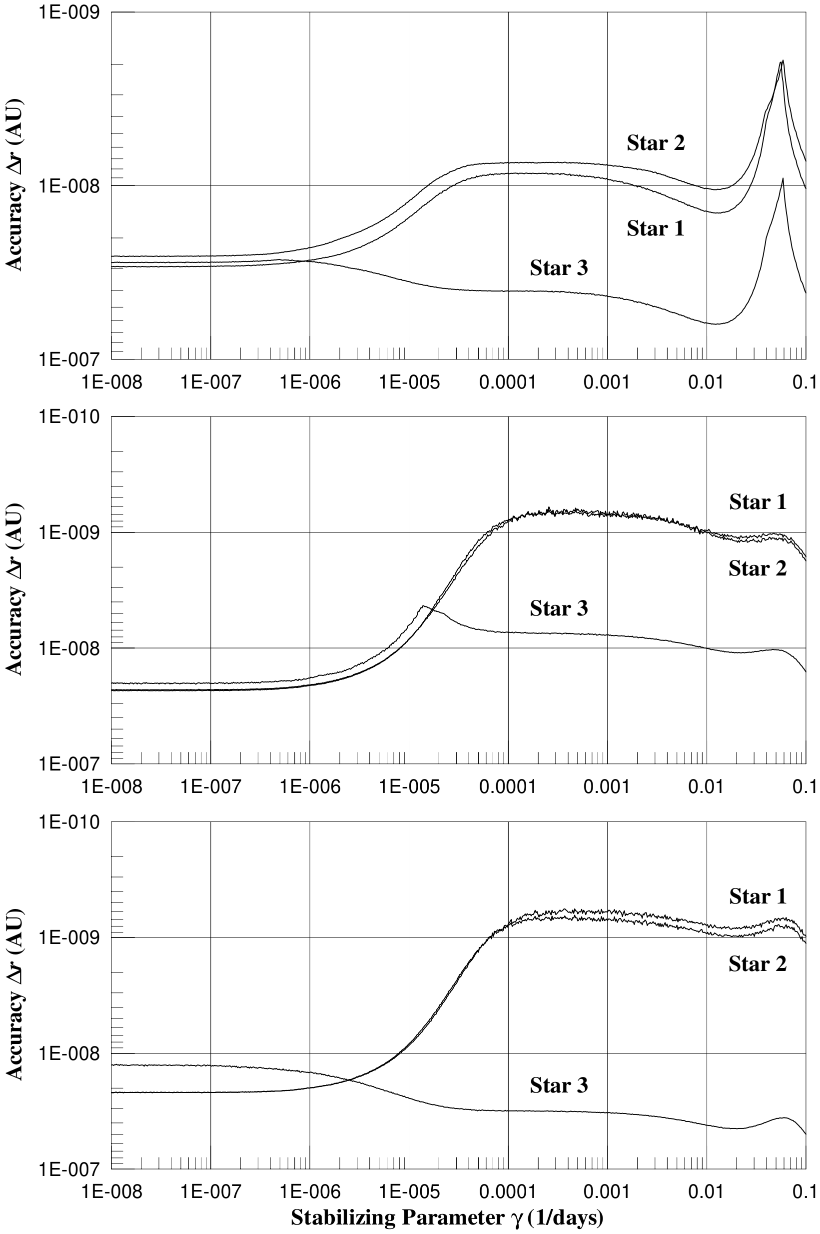

Figure 12. Errors in positions of the stars at different stabilizing parameters. The plots are for $m_3 = 0.2, 0.1, 0.02$ respectively downward.

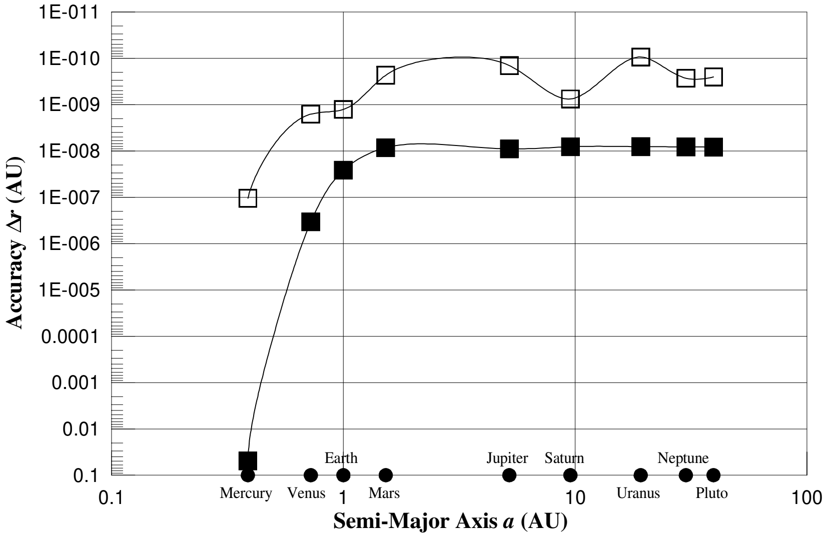

Figure 13. Accuracies of simulating planetary orbits (stabilized — □, non-stabilized — ■)

The figure clearly shows the stabilization works only for the equations of the massive stars and it is quite unable to cardinally improve the accuracy of integration of third star's equations. Also, it is easy to notice that the accuracy characteristic for third star strongly correlates to those for two other stars. As already stated in section 6, it results from that through the stabilizing terms the errors of the two-star dynamical system partly spill over into the numerical orbit of third star as is more evident at $m_3 = 0.02$ when the energy is weakly sensitive to the integration errors of third star's equations.

Anyway, in spite of the increase of accuracy for two massive stars, one cannot regard the stabilization here as an efficient tool with respect to the whole dynamical system. Apparently, a situation like this should occur in other multi-body systems as well.

Finally, we exhibit the efficiency of stabilization in planetary problems by the example of the Solar system. Simulation of planetary orbits has been realized over a time interval 10000 years within the multi-body problem while for every orbit the stabilization of the perturbed two-body problem has been employed. As stabilizing parameters the mean motions of the planets have been taken. The accuracy of integrating ($L = 6$) both stabilized and non-stabilized equations of motion is presented in fig. 13.

As is expected, the stabilization turns out to be the most efficient for Mercury where it reduces errors by almost 6 orders. However, for the other planets errors are reduced only not more than by 2 orders. Quite low accuracy of stabilized integration for the terrestrial planets and, therefore, low efficiency of the stabilization is very likely to be caused by significant influence of the remaining instability.

10. Conclusions

As we have seen, stabilization shows its exceptional abilities in the satellite problems where objects under consideration obey mainly the influence from the gravitational field of the central planet. For Amalthea it has been displayed that stabilization is able to raise the accuracy of integration almost by 5 orders. As a matter of principle, non-stabilized simulations attain to the same accuracy but due to heavy expenses of computing time. From this point of view, stabilization can also be regarded as a means for raising simulating speed.

Of course, the theoretical innovation never at first conciliates the practician. And we think that stabilization will not too. In fact, the set of differential equations after stabilization get complicated: there appear new terms and one subsidiary equation. However, this complication is really nothing to that of the differential equations which are used in practice. And what is more, rewriting a code for stabilization takes only several tens of minutes from an ordinary programmer.

Acknowledgements

The author expresses his sincere thanks to Prof. Bordovitsyna for valuable remarks and proposals.

The paper has been supported by RFBR Grant 01-02-17266.

References

Ascher, U.M., Chin, H., Petzold, L., Reich, S.: 1995, 'Stabilization of Constrained Mechanical Systems with DAEs and Invariant Manifolds', *J. Mech. Struct. Machines* **23**, 135–138.

Baumgarte, J.: 1972, 'Stabilization of Constraints and Integrals of Motion in Dynamical Systems', *Comp. Math. Appl. Mech. Eng.* **1**, 1–16.

Baumgarte, J.: 1973, 'Numerical Stabilization of the Differential Equations of Keplerian Motion', *Cel. Mech.* **5** (4), 490–501.

Bordovitsyna, T.V., Sukhoplyueva, L.E.: 1980, 'Research in the Efficiency of Numerical Algorithms Using Stabilizing Transformations', *Bull. ITA AS USSR* **10** (163), Leningrad: ITA AS USSR, 591–596.

Burdet, C.A.: 1968, 'Theory of Kepler Motion: the General Perturbed Two Body Problem', *Z. Angew. Meth. Phys.* **19**, 345–368.

Chin, H.: 1995, *Stabilization Methods for Simulations of Constrained Multibody Dynamics*, PhD thesis, Institute of Applied Mathematics, University of British Columbia, Canada.

Everhart, E.: 1974, 'Implicit Single Sequence Methods for Integrating Orbits', *Cel. Mech.* **10**, 35–55.

Jacobson, R.A.: 1994, 'Revised Ephemerides of the Inner Jovian Satellites', *JPL IOM* **314**, 10–101.

Lieske, J.H.: 1998, 'Galilean Satellites Ephemerides E-5', *Astron. Astrophys.* **129**, 205–217.

Murison, M.A.: 1989, 'On an Efficient and Accurate Method to Integrate Restricted Three-Body Orbits', *Astron. J.* **97** (5), 1496–1509.

Nacozy, P.E.: 1971, 'The Use of Integrals in Numerical Integrations of the N-Body Problem', *Astrophys. Space Sci.* **14**, 40–51.

Rohde, J.R., Sinclair, A.: 1992, 'Orbital Ephemeredes and Rings of Satellites', *Explanatory Supplement to the Astronomical Almanac*, 325–381.

Stiefel, E.L., Scheifele, G.: 1971, *Linear and Regular Celestial Mechanics*. Springer-Verlag, Berlin, Heidelberg, New York.