

A New Method for the Statistical Simulation of the Virtual Values of Parameters in Inverse Orbital Dynamics Problems

V. A. Avdyushev

Research Institute of Applied Mathematics and Mechanics, Tomsk State University, pr. Lenina 36, Tomsk, 634050 Russia

Received May 6, 2008

Abstract—A Monte Carlo-type method for simulating virtual values of the parameters in inverse orbital dynamics problems for highly nonlinear cases is proposed. The method is based on imitating Fisher's statistics employed to specify the confidence region, and is implemented by solving repeatedly nonlinear least-squares problems with various samples of simulated observations obtainable by suitable random variations.

PACS numbers: 95.10.Ce, 95.10.Eg, 02.50.-r

DOI: 10.1134/S0038094609060094

INTRODUCTION

With the advent of quick-operating and multiprocessor computers, lately statistic simulation of possible parameter values gets employer often for investigating uncertainties in orbits determined from observations the statistical simulation of the virtual values of orbital parameters has become increasingly useful (Chernitsov et al., 1998; Milani, 1999; Virtanen et al., 2001; Bodrovitsyna et al., 2001; Williams et al., 2005; Muinonen et al., 2006; Avdyushev and Banschikova, 2007); this is generally useful in planning future observations, the identification of celestial bodies, and problems regarding asteroid hazard.

A set of virtual values whose density answers the probability density of the true values of orbital parameters is usually constructed starting by the use of estimations of the linearized least-squares problem (LS). However, the connection between the representation of observations and the parameters being in general nonlinear, the involvement of linear estimations for an LS problem for the statistical simulation of virtual values is legitimate, provided only these values cover a sufficiently small region in the parametric space (in which the above connection may admit a good linear approximation). Otherwise, linear estimations will be incapable of giving a reliable estimate of a stochastic scatter.

In the paper are discussed different methods designed for the statistic simulation of possible parameter values. A new method is proposed for building an extensive spread of virtual values when the nonlinearity of an orbital model is considerable and it should not be neglected. The method is based on imitating Fisher's statistics employed to specify the confidence region (Draper, Smith, 1981) and it is realized by repeatedly solving a nonlinear LS problem with various sampled of pseudoobservations generated by appropriate random varying the orbital model representations of real

observations. Moreover, the influence of the nonlinearity caused by intrinsic curvature of estimation space (Bates Watts, 1980; Draper, Smith, 1981) on the results of simulating possible parameter values is numerically explored and also the efficiency of all the considered methods is investigated, especially in the case of small samples of observations.

THE LEAST-SQUARES PROBLEM AND CONFIDENCE REGIONS

We want to solve the overdetermined system of equations

$$\mathbf{P}^O = \mathbf{P}^C(\mathbf{q}), \quad (1)$$

where \mathbf{q} is a K -dimensional vector of parameters to be determined, \mathbf{P}^O is an N -dimensional vector of measurable values (which involves errors $\delta\mathbf{P}^O$), and \mathbf{P}^C is a given N -dimensional vector function of parameters \mathbf{q} . The errors $\delta\mathbf{P}^O$ are assumed to be random, Gaussian, mutually independent, of zero mean, and with variance s^2 . In this case, the best estimations $\hat{\mathbf{q}}$ of the parameters are obtainable from the least-squares principle:

$$S(\hat{\mathbf{q}}) = \|\mathbf{P}^O - \mathbf{P}^C(\hat{\mathbf{q}})\|^2 \longrightarrow \min. \quad (2)$$

In the linear case,

$$\mathbf{P}^C(\mathbf{q}) = \mathbf{A}\mathbf{q}, \quad (3)$$

where \mathbf{A} is a constant rectangular $N \times K$ matrix. Then, estimation $\hat{\mathbf{q}}$, which minimizes $S(2)$, is as follows:

$$\hat{\mathbf{q}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{P}^O. \quad (4)$$

In the nonlinear case, the Gauss–Newton iterative method

$$\mathbf{q}_{k+1} = \mathbf{q}_k - \frac{1}{2} \left[\mathbf{Q}^{-1} \frac{\partial S}{\partial \mathbf{q}} \right] (\mathbf{q}_k), \tag{5}$$

$$\mathbf{Q} = \left[\frac{\partial \mathbf{P}^C}{\partial \mathbf{q}} \right]^T \frac{\partial \mathbf{P}^C}{\partial \mathbf{q}} \quad (k = 0, 1, 2, \dots).$$

is generally useful in finding estimation $\hat{\mathbf{q}}$.

The simulation of virtual values of the parameters is closely bound with the notion of a confidence region. In linear problem (3), a $100\alpha\%$ confidence region in the space of estimated parameters \mathbf{q} is, by definition, an ellipsoidal set, with the center $\hat{\mathbf{q}}$, which contains the true solution $\bar{\mathbf{q}}$ (obtainable with precise measurements of $\bar{\mathbf{P}}$, i.e., when $\delta\mathbf{P}^0 = 0$) with probability α . Formally, a confidence region is given as follows (Draper, Smith, 1981):

$$S(\mathbf{q}) - S(\hat{\mathbf{q}}) = (\mathbf{q} - \hat{\mathbf{q}})^T \mathbf{A}^T \mathbf{A} (\mathbf{q} - \hat{\mathbf{q}}) \leq K \sigma^2 F(K, N - K, \alpha) = (\kappa_\alpha \sigma)^2, \tag{6}$$

here, $\sigma^2 = S(\hat{\mathbf{q}})/(N - K)$ is an unbiased estimation for s^2 , $F(K, N - K, \alpha)$ is an α -quantile of a Fisher probability function with degrees of freedom K and $N - K$, and $\kappa_\alpha = \sqrt{KF}$.

By (6), the random vector \mathbf{q} is asymptotically normal in K -dimensional space, the density function being

$$f(\mathbf{q}) = \frac{1}{\sqrt{(2\pi)^K \det \mathbf{C}}} \exp\left(-\frac{1}{2}(\mathbf{q} - \hat{\mathbf{q}})^T \mathbf{C}^{-1}(\mathbf{q} - \hat{\mathbf{q}})\right), \tag{7}$$

where

$$\mathbf{C} = \sigma^2 (\mathbf{A}^T \mathbf{A})^{-1} \tag{8}$$

is a $K \times K$ covariance matrix of \mathbf{q} .

METHOD OF DISTURBED OBSERVATIONS

There are several methods¹ to simulate the virtual values of the parameters with probability density (7). We feel, however, that the most simple approach should involve the matrix (factor) $\mathbf{C}^{1/2}$ of the Cholesky decomposition $\mathbf{C}^{1/2}(\mathbf{C}^{1/2})^T = \mathbf{C}$. Such a matrix exists and is unique, since the covariance matrix (8) is symmetric and positive definite.

An algorithm for simulating the virtual values of the parameters with the involvement of a Cholesky matrix is based on the formula (Aivazyan et al., 1983; Andronov et al., 2004)

$$\mathbf{q} = \hat{\mathbf{q}} + \mathbf{C}^{1/2} \boldsymbol{\eta}, \tag{9}$$

here, $\boldsymbol{\eta}$ is a K -dimensional normally distributed random vector with zero expectation and unit variance. One can

¹ See, e.g., (Chernitsov et al., 1998; Bordovitsyna, 2001; Muinonen, 2006).

easily verify that vector \mathbf{q} is normally distributed with probability density (7). The repeated substitution of standard multivariate normally distributed random vectors $\boldsymbol{\eta}$ into formula (9) gives a finite set of virtual values of the parameters of \mathbf{q} as a discrete representation of probability density (7).

It is worth noting that the stochastic scatter of \mathbf{q} can be simulated using (4) by the repeated introduction of normally distributed unbiased vector (perturbation) $\delta\mathbf{P}$ with variance σ^2 into \mathbf{P}^0 ; i.e., by the formula

$$\mathbf{q} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T (\mathbf{P}^0 + \delta\mathbf{P}) = \hat{\mathbf{q}} + (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \delta\mathbf{P}. \tag{10}$$

One may readily check that in this case, the scatter of vector \mathbf{q} will also correspond to the probability density (7), and in fact, by (10), may be thought of as representing the set of solutions of a liner least-squares problem with various samples of disturbed observations $\mathbf{P}^0 + \delta\mathbf{P}$.

In the nonlinear case, if errors $\delta\mathbf{P}^0$ are sufficiently small and thus incur small parametric errors in \mathbf{q} , then LS estimations may be considered as far as the linear problem (3) is concerned; therefore, in order to simulate the virtual values of the parameters, we may invoke algorithm (9), where, however, the Cholesky matrix $\mathbf{C}^{1/2}$ is calculated for the covariance matrix given in estimation $\hat{\mathbf{q}}$:

$$\mathbf{C} = \sigma^2 \mathbf{Q}^{-1}. \tag{11}$$

Unfortunately, in the highly nonlinear case, when the scatter of parameters in \mathbf{q} is sufficiently large, the utilization of algorithm (9) is not valid, inasmuch as the confidence regions may be distinctly different from the ellipsoidal regions.

Assume now that there exists a system K of parameters \mathbf{r} , relative to which the LS problem becomes linear; moreover, parameters \mathbf{q} and \mathbf{r} being in one-to-one correspondence: $\mathbf{q} = \mathbf{T}(\mathbf{r})$. Further, we obtain the set of LS solutions of (1) in the spaces of \mathbf{q} and \mathbf{r} with various samples $\mathbf{P}^0 + \delta\mathbf{P}$. It is readily verified that the estimations $\hat{\mathbf{q}}$ and $\hat{\mathbf{r}}$, which minimize objective functions $S(\mathbf{q})$ and $S(\mathbf{T}(\mathbf{r}))$ with the same sample $\mathbf{P}^0 + \delta\mathbf{P}$, are connected by the relation $\hat{\mathbf{q}} = \mathbf{T}(\hat{\mathbf{r}})$. Consequently, the LS scatter in the space of \mathbf{q} an image of the LS scatter in the space of \mathbf{r} and vice versa. In addition to that, the scatter of LS solutions in the space of \mathbf{r} adequately represents the set of virtual values of the parameters. Hence, the LS scatter in the space of \mathbf{q} will also provide an adequate representation of the set of virtual values of the parameters, but now with the probability density

$$f(\mathbf{q}(\mathbf{r})) = \frac{1}{\sqrt{(2\pi)^K \det \mathbf{C}_r}} \exp\left(-\frac{1}{2}(\mathbf{r} - \hat{\mathbf{r}})^T \mathbf{C}_r^{-1}(\mathbf{r} - \hat{\mathbf{r}})\right) \left| \frac{\partial \mathbf{q}}{\partial \mathbf{r}} \right|,$$

where \mathbf{C}_r is the covariance matrix of \mathbf{r} ; also, $|\partial \mathbf{q} / \partial \mathbf{r}|$ should be viewed as the ratio of the corresponding elementary volumes in the spaces of \mathbf{q} and \mathbf{r} .

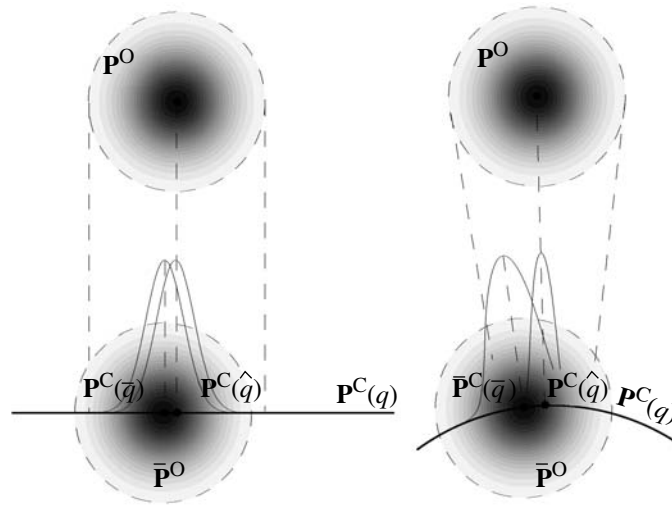


Fig. 1. Distribution of densities (bell curves) of the orthogonal projection of the sets of possible measurements onto the estimation subspace in the flat (left-hand part) and nonflat (right-hand part) cases. For purposes of clarity, the situation with the position of the sets is shown as exaggerated; in fact, in the space of the measurements they are located much nearer to each other, the significant parts overlapping.

Thus, in the nonlinear case, the set of virtual values of the parameters may be simulated by solving repetitively the LS problem with various artificially produced samples of observations², $\mathbf{P}^O + \delta\mathbf{P}$:

$$\mathbf{q} : S(\mathbf{q}) = \|\mathbf{P}^O + \delta\mathbf{P} - \mathbf{P}^C(\mathbf{q})\|^2 \rightarrow \min. \quad (12)$$

It worth pointing out that this approach does not involve the passage to the parameters \mathbf{r} . However, for its applicability we need to know that such a system of parameters exists. On the construction of such sets of the parameters relative to that the problem became either linear or weakly nonlinear, one can find, for example, in citep (Beale, 1960; Hartley, 1964).

One obvious deficiency in the simulation of the LS scatter is the repeated iterative solution of the nonlinear least-squares problem (5), which sometimes may involve too much computer time. For example, in the example below it took about one day to construct the set of all virtual values of the parameters (like the LS scatter over 10000 solutions) using an Intel® Pentium® Dual-Core 3GHz processor.

METHOD OF DISTURBED ESTIMATIONS

According to the geometric interpretation given in (Bates, Watts, 1980; Draper, Smith, 1981), a point $\mathbf{P}^C = \mathbf{P}^C(\hat{\mathbf{q}})$ in an N -dimensional observation space \mathbf{P} is an orthogonal projection of point \mathbf{P}^O in the K -dimensional estimation subspace $\mathbf{P}^C(\mathbf{q})$ (Bates, Watts, 1980; Draper, Smith, 1981). Consequently, the set of virtual values of

the parameters \mathbf{q} , viewed as the scatter of LS solutions (12), can be thought of as carrying into the parametric space \mathbf{q} the set of points \mathbf{P}^C obtained by the orthogonal projection of the set of all possible measurements $\mathbf{P}^O + \delta\mathbf{P}$ into the estimation subspace \mathbf{P}^C .

By (6), a confidence region in the linear case is the set of points \mathbf{P}^C satisfying

$$\|\mathbf{P}^C - \hat{\mathbf{P}}^C\|^2 \leq (\kappa_\alpha \sigma)^2. \quad (13)$$

Assigning the confidence region to the center $\bar{\mathbf{P}}^C = \mathbf{P}^C(\bar{\mathbf{q}}) = \bar{\mathbf{P}}^O$, we obtain a certain spherical region

$$\|\mathbf{P}^C - \bar{\mathbf{P}}^C\|^2 \leq (\kappa_\alpha \sigma)^2, \quad (14)$$

which contains, with probability α , an approximative solution $\hat{\mathbf{P}}^C = \mathbf{P}^C(\hat{\mathbf{q}})$. From (13) and (14), it follows that the differences $\mathbf{P}^C - \hat{\mathbf{P}}^C$ and $\mathbf{P}^C - \bar{\mathbf{P}}^C$ have a K -dimensional normal distribution with the same variance. This agreement in the distribution of probability densities in simulating the virtual values of the parameters is necessary, and it takes place in the linear case. In fact, subspace $\mathbf{P}^C(\mathbf{q})$ being flat, the distribution of estimations \mathbf{P}^C (viewed as projections of possible measurements $\mathbf{P}^O + \delta\mathbf{P}$ with the same scatter $\delta\mathbf{P}$) is conserved irrespective of the position of \mathbf{P}^O (Fig. 1).

At the same time, if the estimation subspace $\mathbf{P}^C(\mathbf{q})$ is not flat, then this situation does not hold in general, because, as distinct from the flat case, the set of measurements $\mathbf{P}^O + \delta\mathbf{P}$, after being projected into the subspace, has a density which is dependent on the distance from \mathbf{P}^O to the subspace: the projection becomes more

² This nonlinear evaluation of parametric accuracy, called the Observational Monte Carlo method, had been implemented (however, without any justification) in the software Find_Orb, obtainable from the website www.projectpluto.com

dense as \mathbf{P}^0 moves away from the center of the curvature of $\mathbf{P}^C(\mathbf{q})$ and vice versa (Fig. 1).

In order to compensate for the effect arising from the misalignment of probability densities when simulating the virtual values of the parameters, it will be convenient to employ a modification of scheme (12):

$$\mathbf{q} : S(\mathbf{q}) = \|\hat{\mathbf{P}}^C + \delta\mathbf{P} - \mathbf{P}^C(\mathbf{q})\|^2 \longrightarrow \min, \quad (15)$$

Here, the measurement vector \mathbf{P}^0 is replaced by its projection $\hat{\mathbf{P}}^C$. One may readily show that in the linear case, the modified scheme produces, like (9) and (10), a normally distributed vector \mathbf{q} with probability density (7). This scheme will be most efficient if the curvature of space $\mathbf{P}^C(\mathbf{q})$ is constant at least near the projection of the set.

Scheme (15) is frequently useful in practice for simulating the virtual values of the parameters (Rohlf's et al., 1986; Press et al., 1987; Dambis et al., 1995; Nikiforov, 1999); however, it is never thought of as a means for tackling problems involving nonlinearities due to the internal curvature of the estimation subspace (Bates, Watts, 1980; Draper, Smith, 1981).

Further, it should be noted that even in the linear case the use of the method of disturbed observations (or estimations), like the approach involving the Cholesky matrices, gives only an approximate representation of the virtual values of the parameters, the representation becoming worse as the number N of measurements becomes smaller. In fact, it is assumed in schemes (9) and (10) that vector \mathbf{q} has a Gaussian distribution; whereas, according to the definition of the confidence region (6), the parametric vector \mathbf{q} is only asymptotically normal. In this connection, in order to improve the simulation of virtual values of the parameters, we put forward below simple modifications of schemes (9) and (10), which, however, have considerable practical importance.

First, let us recall that in the linear case the notion of a confidence region is given from the knowledge that the random variable

$$U = \frac{S(\mathbf{q}) - S(\hat{\mathbf{q}})}{K\sigma^2} \quad (16)$$

(with respect to $\mathbf{q} = \bar{\mathbf{q}}$ and with various samples of observations with all possible normal errors) is itself Fisher distributed with degrees of freedom K and $N-K$; i.e.,

$$U = F(K, N-K) = \frac{N-K}{K} \frac{\chi_K^2}{\chi_{N-K}^2}, \quad (17)$$

where χ_K^2 and χ_{N-K}^2 are random variables distributed as χ^2 squared with degrees of freedom K and, respectively, $N-K$. Consequently, the set of virtual values of \mathbf{q} with respect to a fixed estimation $\hat{\mathbf{q}}$ should be con-

structed so as to ensure that U would be also Fisher distributed with the same degrees of freedom.

In order that scheme (9) be capable of modeling this vector \mathbf{q} , it is sufficient to introduce the random value χ_{N-K}^2 as follows:

$$\mathbf{q} = \hat{\mathbf{q}} + \mathbf{C}^{1/2} \boldsymbol{\eta} \sqrt{\frac{N-K}{\chi_{N-K}^2}}. \quad (18)$$

In fact, applying (18) gives

$$(\mathbf{C}^{1/2})^{-1}(\mathbf{q} - \hat{\mathbf{q}}) = \boldsymbol{\eta} \sqrt{\frac{N-K}{\chi_{N-K}^2}}.$$

Calculating the inner products of its left and right-hand sides and invoking (8), we obtain

$$\frac{(\mathbf{q} - \hat{\mathbf{q}})^T \mathbf{A}^T \mathbf{A} (\mathbf{q} - \hat{\mathbf{q}})}{\sigma^2} (= KU) = (N-K) \frac{\chi_K^2}{\chi_{N-K}^2}.$$

Hence, by (17), $U = F(K, N-K)$.

A similar improvement applies to scheme (10), giving

$$\mathbf{q} = \hat{\mathbf{q}} + (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \delta\mathbf{P} \sqrt{\frac{N-K}{\chi_{N-K}^2}}. \quad (19)$$

Hence, by (19),

$$\mathbf{A}(\mathbf{q} - \hat{\mathbf{q}}) = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \delta\mathbf{P} \sqrt{\frac{N-K}{\chi_{N-K}^2}}.$$

Since $\mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \delta\mathbf{P} = \boldsymbol{\sigma}\boldsymbol{\eta}$, calculating the inner product of left and right-hand sides and solving it for U (from (16)), we infer that this variable, as before, is Fisher distributed with degrees of freedom K and $N-K$.

Thus, extending scheme (19) to the nonlinear case, we obtain

$$\mathbf{q} : S(\mathbf{q}) = \|\mathbf{P}^C + \delta\mathbf{P}r - \mathbf{P}^C(\mathbf{q})\|^2 \longrightarrow \min, \quad (20)$$

where $r = \sqrt{(N-K)/\chi_{N-K}^2}$. Clearly, since scheme (20) degenerates into scheme (15) as the number of observations N increases, we may assume that the comparative effectiveness of the first will be greater the smaller number N is.

A modulation factor r multiplying perturbations $\delta\mathbf{P}$ is, in essence, necessary for accommodating the fact that variable σ^2 , regarded as a statistical estimation of the variance of observation errors s^2 and obtainable from the sample from observed data with random errors, is itself a random variable. If observation errors are normal, then, in the linear case,

$$\sigma^2 = s^2 \frac{\chi_{N-K}^2}{N-K}.$$

In the nonlinear (nonflat) case, the distribution of σ^2 is unknown in general. In order to compensate for the

absence of information about the distribution of σ^2 , we suggest the simulation of virtual values of the parameters be held at two stages. The first stage involves the preliminary use of scheme (15), where samples of random perturbations $\delta\mathbf{P}$ are simulated with mean square error σ^2 and are retained together with the corresponding estimations σ_*^2 , obtainable via the inverse problem. Thus, we gather information about the distribution of σ^2/s^2 assuming that it is the same as for σ_*^2/σ^2 . Then, at the second stage, we perform the statistic simulation by scheme (20) with the involvement of the retained samples and the factors $r = \sigma/\sigma_*$. Clearly, in the flat case, the two-phase approach produces the same results as scheme (20) with the original multiplier r .

NUMERICAL EXAMPLE

The above algorithms for simulating the virtual values of the parameters were examined in the context of a highly nonlinear inverse problem in satellite dynamics. For the newly discovered Jovian satellite S/2003 J04, LS estimations of orbital parameters \mathbf{q} (which we took to be six components of the dynamic state vector (position and velocity) $\mathbf{q} = (\mathbf{x}_0, \dot{\mathbf{x}}_0)^T$ at epoch t_0) were determined based on meager observations (angular coordinates) \mathbf{P}^O (Emel'yanov et al., 2006) and by the use of high-precision simulation of its motion $\mathbf{P}^C(\mathbf{q})$ (Avdyushev and Banschikova, 2007), the mean square error being $\sigma = 0.20''$.

The observation of the satellite covered only a short orbital arc (11 observations during several dozens of days), and so, even with the initial epoch t_0 lying inside the observation interval, the probability regions turn out to be extremely large (Avdyushev and Banschikova, 2007) and, hence, the nonlinearity effect becomes substantial.

We took as true observations $\bar{\mathbf{q}}$ the LS estimations of orbital parameters determined by real observations, and based on these data we simulated precise observations $\bar{\mathbf{P}}^O = \mathbf{P}^C(\bar{\mathbf{q}})$ of the satellite at real instances of observations. Then, we introduced $\bar{\mathbf{P}}^O$ to the sample of normal unbiased errors $\delta\mathbf{P}^O$ with variance $s^2 = \sigma^2$, and by observations $\bar{\mathbf{P}}^O + \delta\mathbf{P}^O$ we obtained the LS estimation for $\hat{\mathbf{q}}$, for which the mean square error was found to be $\sigma = 0.21''$. Finally, using algorithms (9) and (15), the set of virtual values of the parameters with respect to estimation $\hat{\mathbf{q}}$ was constructed; each set was represented by 10000 solutions. In order to sharpen the nonlinearity effect, we deliberately removed the initial epoch t_0 from the observation time interval, its distance from the epoch of observations was in the order of one orbital period of the satellite.

We note that in the construction of the set of virtual values of the parameters via scheme (15), we took estimation $\hat{\mathbf{q}}$ as the initial approximation \mathbf{q}_0 in the Gauss–Newton method (5); however, since the region covered with LS solutions is vast, such an approximation does not always yield the convergence of the iterative process. In this case, use was made of the damped Gauss–Newton method; that is, the correction factor in scheme (5) was reduced by a factor of 100. Despite the appreciable retardation in the iterative process, it was capable of providing convergence for any samples of observations.

Figure 2 shows the simulation results, in jovicentric coordinates, of various orbital parameters of the satellite via different algorithms. Clearly, the distributions of possible initial positions of the satellite give us a useful indication of the sizes of confidence regions of both linearized and nonlinear problems corresponding to a probability in the order of 99.99%.

Thus, as is apparent from the picture, both probability regions are extremely vast. The region of the linearized problem is an elongated ellipsoid; however, it does not cover the true solution (unlike the region representable by LS solutions (15)).

It is worth pointing out that the probability regions intersect only near the estimation $\hat{\mathbf{q}}$, and, hence, algorithm (9) is believed to be justified provided only the probability regions are considerably smaller (down to several dozen or more).

It is also interesting to note that the density of the distribution of $U(\mathbf{q})$ (from (16)), obtainable through all possible orbital parameters \mathbf{q} (from (15)), is close (up to the scale parameter $K^{-1}\sigma^2$) to the probability density function χ_6^2 with degree of freedom $K = 6$ (Fig. 3). In this case, the χ -square test, as calculated on the interval in question, with partition number 100, was found to be $\chi^2 \approx 87$, which is considerably smaller than the standard 95% quantile: $\chi_{95\%}^2 \approx 123$. In the linear case, $U(\mathbf{q})$ is exactly χ_6^2 ; whereas its distribution with possible parameters (9) has absolutely nothing to do with distribution χ_6^2 .

As we have already observed, the simulation of the virtual values of the parameters by the use of a repeated solution of the LS problem is justified merely by the existence of a system of parameters \mathbf{r} , with respect to which the LS problem becomes linear. Such a system of parameters can be produced if the K -dimensional subspace defined by $\mathbf{P}^C(\mathbf{q})$ in the N -dimensional subspace of measurable values \mathbf{P} is flat (Beale, 1960; Bates, Watts, 1980), i.e., when each point of this subspace is given in some Cartesian system of coordinates containing K basis vectors. In addition to this, the coordinates of this system may be taken as parameters \mathbf{r} .

In the nonlinear LS problem, the estimation subspace $\mathbf{P}^C(\mathbf{q})$ is, in general, nonflat, and a system of

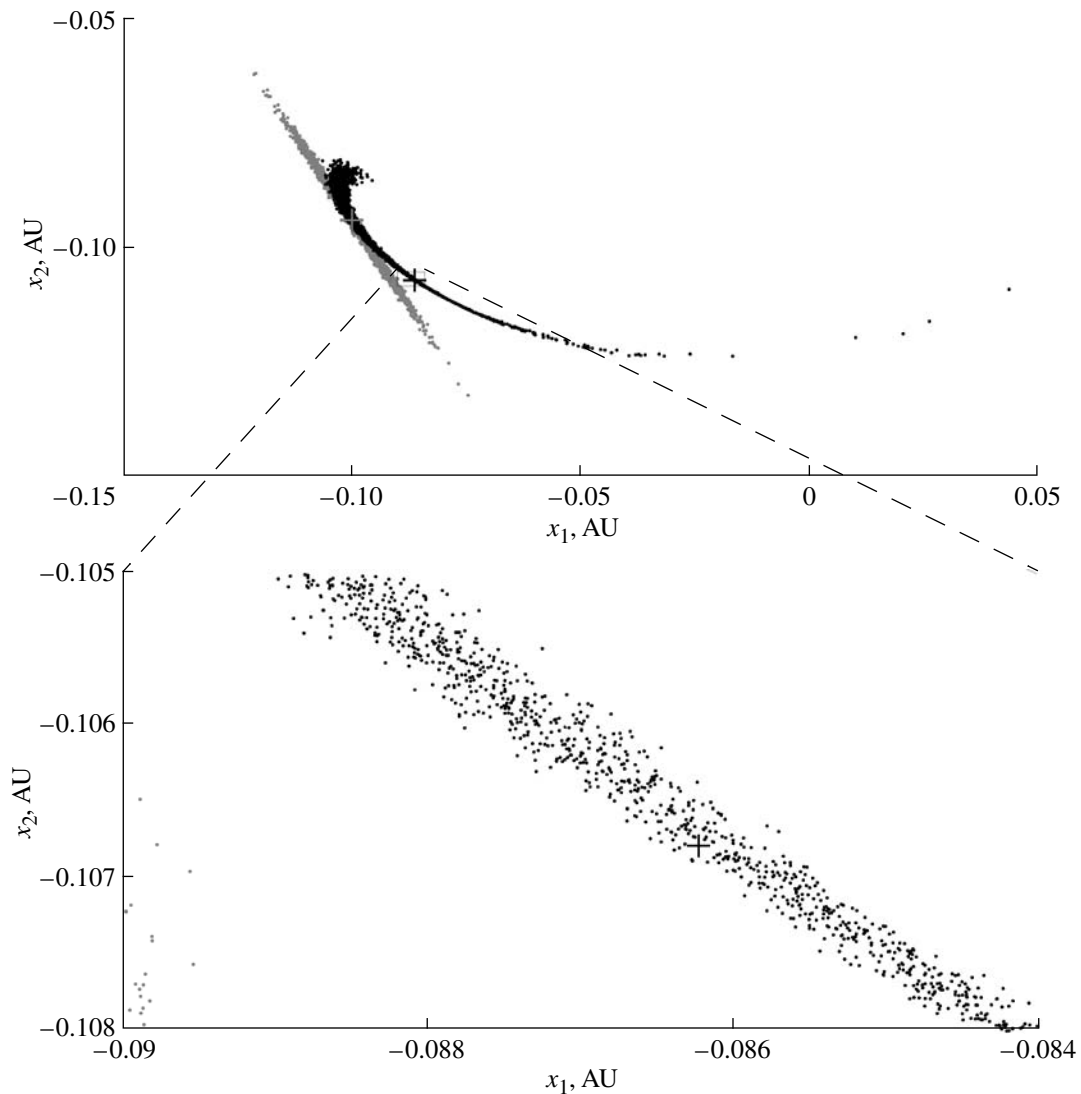


Fig. 2. Possible positions of satellite S/2003 J04 on the coordinate plane as constructed by algorithms (9) (gray dots) and (15) (black dots) near the true solution in various scales. The projection of the true solution $\bar{\mathbf{q}}$ is black checked and the projection of the estimation $\hat{\mathbf{q}}$ is gray checked.

parameters \mathbf{r} , which linearizes the problem, may fail to exist. Nevertheless, if the subspace differs nonessentially from a flat one (in the range of estimations of interest), then the method of disturbed observation is quite practical for simulating virtual values of the parameters.

Figure 4 indicates the distribution of angular deviations φ of the subspace $\mathbf{P}^C(\mathbf{q})$ at points of virtual values of \mathbf{q} from the tangent space in the LS estimation $\hat{\mathbf{q}}$ of the flat subspace. Angle φ is given by the formula

$$\cos(90^\circ - \varphi) = \frac{(\mathbf{P}^0 - \hat{\mathbf{P}}^C)(\mathbf{P}^C(\mathbf{q}) - \hat{\mathbf{P}}^C)}{\|\mathbf{P}^0 - \hat{\mathbf{P}}^C\| \|\mathbf{P}^C(\mathbf{q}) - \hat{\mathbf{P}}^C\|},$$

where $\hat{\mathbf{P}}^C = \mathbf{P}^C(\hat{\mathbf{q}})$. Clearly, for a flat subspace we have $\varphi = 0$, because the vectors $\mathbf{P}^0 - \hat{\mathbf{P}}^C$ and $\mathbf{P}^C(\mathbf{q}) - \hat{\mathbf{P}}^C$ are orthogonal (Draper, Smith, 1981; Bates, Watts, 1988).

It is evident from the figure that the values of φ are sufficiently small and most of them are majorized by 0.1° . Consequently, the strong nonlinearity of the problem is mostly due to the unfortunate choice of estimated parameters (Bates, Watts, 1980; 1988). We note that this measure of the so-called intrinsic nonlinearity was put forward earlier in (Beale, 1960), and in fact it can be expressed in terms of angle φ , namely, $\sin^2\varphi$ (Bates, Watts, 1980). In any case, we may assume that the above set of LS solutions does represent well the distribution of virtual values of orbital parameters.

Thus, for the legitimate employment of scheme (12) in nonlinear problems, it is necessary to have some idea of how big the deviation of subspace $\mathbf{P}^C(\mathbf{q})$ is from a flat one; this can be achieved, in particular, by the above method.

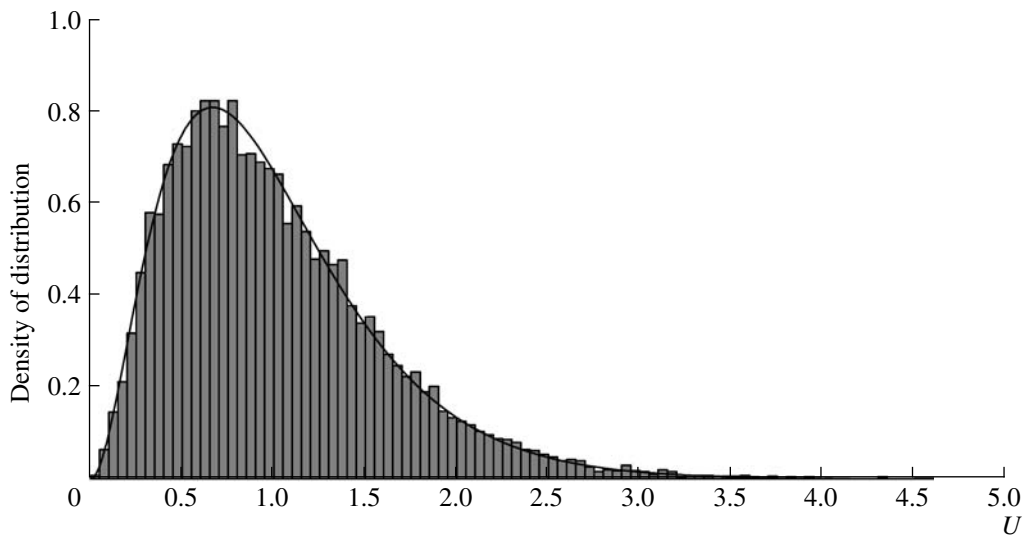


Fig. 3. Distribution of values $U(16)$ as obtained by possible orbital parameters \mathbf{q} (15). The solid curve gives the scaled function of the probability density χ_6^2 with degree of freedom $K = 6$.

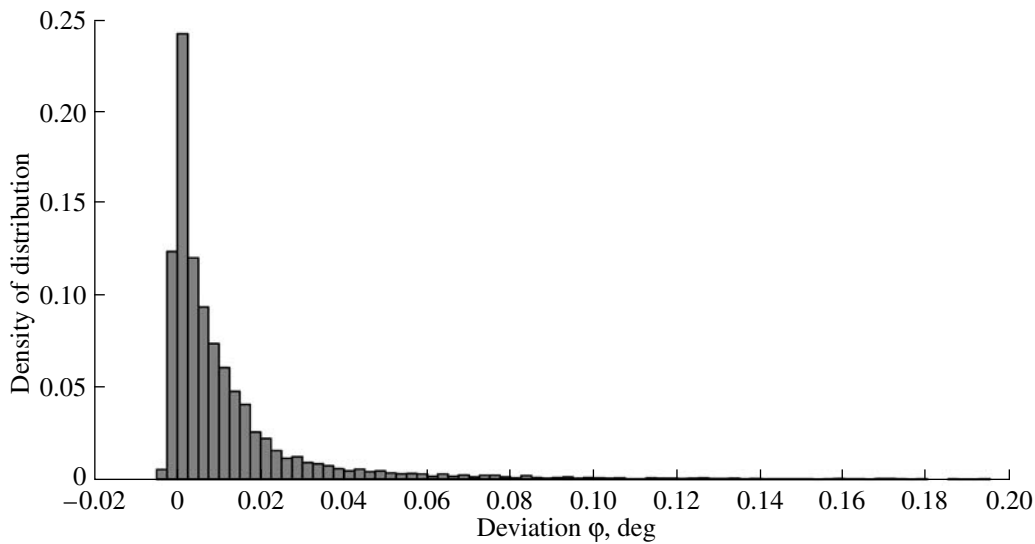


Fig. 4. Distribution of deviations of subspace $\mathbf{P}^C(\mathbf{q})$ at the points of the region of virtual values of the parameters from the tangent subspace in the LS estimation $\hat{\mathbf{q}}$ of the flat subspace.

As the estimation subspace in the inverse problem is almost flat, it follows that the sets of virtual values obtainable via schemes (12) and (15) lie very close together. Nevertheless, it is apparent from Fig. 5 that the scatter of possible satellite positions, as produced by scheme (15), is considerably smaller than the scatter according to scheme (12). In any event, one should bear in mind that the relation between the sizes of the region to be filled with possible satellite positions may be different: this in fact depends upon the relative position of

the vector of observations \mathbf{P}^O and the estimation subspace $\mathbf{P}^C(\mathbf{q})$, and also upon the degree of its intrinsic curvature.

The modified scheme of the method of disturbed estimations (20) has been examined in the same problem, but with the involvement of only some observations (per four time points ($N = 8$)), which also cover the whole observation period of the satellite. It is with these conditions for the inverse problem (with meager

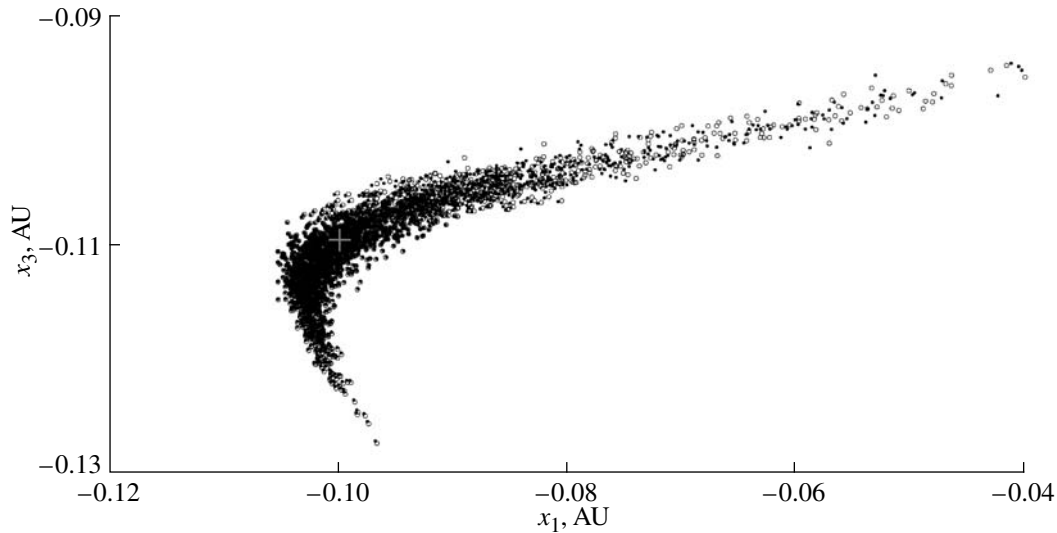


Fig. 5. Possible positions of satellite S/2003 J04 on the coordinate plane as constructed by algorithms (12) (circles) and (15) (points).

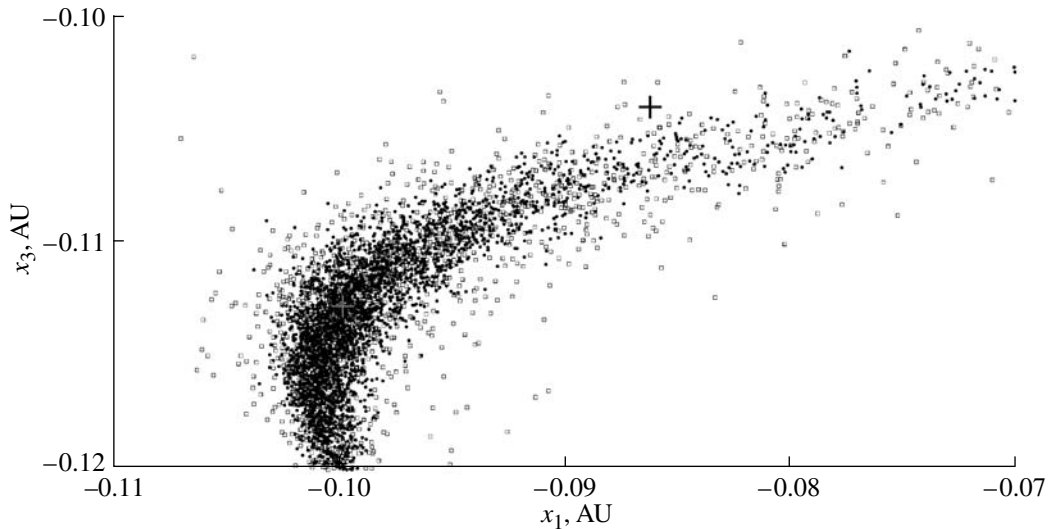


Fig. 6. Possible positions of satellite S/2003 J04 on the coordinate plane as constructed by algorithms (15) (points) and (20) (small squares).

observational data) that one may expect the manifestation of the high effectiveness of scheme (20).

As before, given a sample of artificially produced observations $\bar{\mathbf{P}}^0 + \delta\mathbf{P}^0$ of size $N = 8$, we obtained a new estimation $\hat{\mathbf{q}}$, for which the mean square error was $\sigma = 0.13''$. Then, by schemes (15) and (20), we simulated the possible positions of the satellite, the estimation σ being used for the simulation of the scatter of perturbations $\delta\mathbf{P}$. The results are given in Fig. 6.

It is apparent from the figure that the sets of virtual values, as constructed by different schemes, are markedly different from each other and describe different

distributions of the probability density in the coordinate space. Also, at the point of the true position of the satellite, the probability density via scheme (15) turns out to be significantly smaller than that via scheme (20). At the same time, if the scatter of virtual values is used to indicate the size of the corresponding confidence region, then we may say that scheme (15) represents a confidence region which is significantly smaller than that represented by scheme (20), and the latter, unlike the first, provides for a confident covering of the true values of the parameters. Taken all together, this clearly lends strong support to modified scheme (20)—this being more efficient from the viewpoint of the statistical simulation of virtual values of the parameters.

CONCLUSIONS

The paper puts forward a new method for simulating the virtual values of the orbital parameters in nonlinear inverse problems. The method was tested on a satellite dynamic problem with meager observations, in which, due to the poor accuracy in the determinate orbit, the scatter in the virtual values was very large and, hence, highly distorted by the nonlinearity. The results obtained established that the method in question is more efficient than its analogous methods, especially when the orbit is determined by a meager sample of observations.

ACKNOWLEDGMENTS

The author is grateful to Prof. A.M. Chernitsov for discussion of the results and for valuable remarks and suggestions. This work was supported by the Russian Foundation for Basic Research (project no. 08-02-00359) and by the Ministry of Education of the Russian Federation (project RNP.2.1.1/2629).

REFERENCES

- Avdyushev, V.A., and Banschikova, M.A., Regions of Possible Motions for New Jovian Satellites, *Astron. Vestnik*, 2007, vol. 41, no. 5, pp. 446–452 [*Sol. Syst. Res.* (Engl. Transl.), vol. 41, no. 5, pp. 413–419].
- Aivazyan, S.A., Enyukov, I.S., and Meshalkin, L.D., *Prikladnaya statistika* (Applied Statistics), Moscow: Finansy i Statistika, 1983.
- Andronov, A.M., Kopytov, E.A., and Gringlaz, L.Ya., *Teoriya veroyatnostei i matematicheskaya statistika* (Probability and Mathematical Statistics), St. Petersburg: Piter, 2004.
- Emel'yanov, N.V., Arlo, Zh.Yu., Varfolomeev, M.I., et al., Construction of Theories of Motion, Ephemerides, and Databases for Natural Satellites of Planets, *Kosm. Issled.*, 2006, vol. 44, pp. 136–145 [*Cosmic. Res.* (Engl. Transl.), vol. 44, pp. 128–136].
- Nikiforov I.I., Simulating the Rotation Law of a Flat Subsystem and Determining the Distance to the Galactic Center: Model Feasibility and Complexity Optimization, *Astrofizika*, 1999, Vol. 42, no. 3, pp. 399–406.
- Chernitsov, A.M., Baturin, A.P., and Tamarov, V.A., Analysis of Some Methods of Determining Probabilistic Evolutions of the Motion of Solar System Small Bodies, *Astron. Vestnik*, 1998, vol. 32, no. 5, pp. 459–467.
- Bates D.M., and Watts D.G., Relative Curvature Measures of Nonlinearity, *J. R. Stat. Soc.*, 1980, vol. 42, no. 1, pp. 1–25.
- Bates D.M., and Watts D.G., *Nonlinear Regression Analysis and its Applications*, Wiley, 1988.
- Beale E.M.L., Confidence Regions in Non-Linear Estimation, *J. R. Stat. Soc.*, 1960, vol. 22, no. 1, pp. 41–88.
- Bordovitsyna T., Avdyushev V., and Chernitsov A., New Trends in Numerical Simulation of the Motion of Small Bodies of the Solar System, *Celestial Mech Dyn Astr.*, 2001, vol. 80, no. 3, pp. 227–247.
- Dambis A.K., Mel'nik A.M., and Rastorguev A.S., Rotation Curve of the System of Classical Cepheids and the Distance to the Galactic Center, *Astron. Letters*, 1995, vol. 21, no. 3, pp. 291–307.
- Draper N.R. and Smith H., *Applied Regression Analysis*, Wiley, 1981.
- Hartley H.O., Exact Confidence Regions for the Parameters in Non-Linear Regression Laws, *Biometrika*, 1964, vol. 51, no. 3/4, pp. 347–353.
- Milani A. The Identification Problem I: Recovery of Lost Asteroids, *Icarus*, 1999, vol. 137, pp. 269–292.
- Muinsonen K., Virtanen J., Granvik M., and Laakso T., Asteroid Orbits Using Phase-Space Volumes of Variation, *Mon. Notic Roy. Astron. Soc.*, 2006, vol. 368, pp. 809–818.
- Press W.H., Flannery B.P., Teukolsky S.A., and Vetterling W.T., *Numerical Recipes: The Art of Scientific Computing*, Cambridge: University Press, 1987.
- Rohlf K., Boehme R., Chini R., and Wink J.E., The Rotation Curve of the Galaxy, *Astron. Astrophys.*, 1986, vol. 158, no. 1–2, pp. 181–190.
- Virtanen J., Muinsonen K., and Howell E., Statistical Ranging of Asteroid Orbits, *Icarus*, 2001, vol. 154, pp. 412–431.
- Williams I.P., Ryabova G.O., Baturin A.P., and Chernitsov A.M., Are Asteroid 2003 EH1 and Comet C/1490 Y1 Dynamically Related? *Earth. Moon, and Planets*, 2005, vol. 95, pp. 11–18.