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# SPECIAL PERTURBATION THEORY METHODS IN CELESTIAL MECHANICS. I. PRINCIPLES FOR THE CONSTRUCTION AND SUBSTANTIATION OF THE APPLICATION

**V. A. Avdyushev**

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*The ideas and principles for the construction of methods in special perturbation theory are discussed, and their application to the solution of problems in classical celestial mechanics is substantiated. The problem of short-period perturbations and their effect on the numerical integration is investigated.*

## INTRODUCTION

A variety of formalisms can be used to investigate the motions of celestial bodies. In a numerical simulation of orbits, the formalism represents a system of differential equations which are integrated numerically. In this connection, for the simulation to be efficient, it is of importance to properly formalize the orbital motion since this directly affects the accuracy and speed of the numerical integration.

In rectangular coordinates, the classical equations of an orbital motion around a massive central body can be represented in the form

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{\mu}{|\boldsymbol{x}|^3}\boldsymbol{x} + \boldsymbol{P} \equiv \boldsymbol{F} + \boldsymbol{P}. \tag{1}$$

Here $\boldsymbol{x}$ is the position vector; $t$ is the time; $\mu$ is the gravitational parameter of the central body; $\boldsymbol{F}$ and $\boldsymbol{P}$ are the central and the perturbing force, respectively. We also assume that $|\boldsymbol{P}| \ll |\boldsymbol{F}|$.

The numerical integration of equations (1) encounters the following difficulties. The right sides of equations (1) represent rapidly varying functions which should be integrated with a small step. This increases the body of calculations, and this, in turn, entails rapid accumulation of round-off errors. This difficulty is aggravated with the singularity at $\boldsymbol{x} = \boldsymbol{0}$ present in the equations of motion, which, in the case of a strongly elongated orbit, becomes a reason for an excentric behavior of the right sides. Moreover, the equations are Lyapunov unstable. It is well known that the Lyapunov instability arising in a numerical integration of differential equations may be responsible for the buildup of various errors inevitably accompanying any numerical process.

In the present work we focus our attention on the well-known methods of formalized representation of an orbital motion that eliminate the above difficulties. We shall refer to these methods as methods of special perturbation theory.[1] Using these methods for deriving qualitatively new equations is generally based on some a priori information about the orbital motion under investigation. We state the principles of the methods that involve the solution of the two-body problem on the assumption that the orbit under investigation is near-Keplerian. In particular, we consider linearization and regularization methods, smoothing transformations, the method of variation of coordinates (Encke) and constants, and the Baumgarte–Nacozy numerical stabilization. Note that we give a brief outline of the methods, not pretending to a comprehensive historical excursus. Therefore, we intentionally restrict ourselves to only fundamental, in our opinion, results on the subject of the present work.

[1] In the foreign literature, by special perturbation methods are meant numerical methods [1, 2].

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## 1. LINEARIZATION AND REGULARIZATION

The methods of linearization and regularization are intended to represent the equations of motion in linear and regular form. Let us first consider the equations of a nonperturbed motion

$$\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F}(\boldsymbol{x}). \tag{2}$$

Suppose that equations (2) have integrals

$$\boldsymbol{G}_i(t,\boldsymbol{x},\dot{\boldsymbol{x}}) - \boldsymbol{g}_i \equiv \boldsymbol{0} \quad (i=1,\ldots,n), \quad H_j(t,\boldsymbol{x},\dot{\boldsymbol{x}}) - h_j \equiv 0 \quad (j=1,\ldots,m). \tag{3}$$

Here $\boldsymbol{G}_i$ and $H_j$ are the vector and the scalar integral functions, respectively; $\boldsymbol{g}_i$ and $h_j$ are integral parameters, which are constant for a nonperturbed motion. Introduce a coordinate and a time transform:

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{u}, \quad dt = f(\boldsymbol{x})ds, \tag{4}$$

allowing us to use the new variables $\boldsymbol{u}$ and $s$.

The principal idea of linearization and regularization is to enter the integral relations (3) into the equations written in the new variables [3]. As a result, the equations become [4]

$$\boldsymbol{u}'' = \boldsymbol{A}^{-1}(f^2\boldsymbol{F} + f^{-1}f'(\boldsymbol{A}\boldsymbol{u}' + \boldsymbol{A}'\boldsymbol{u}) - 2\boldsymbol{A}'\boldsymbol{u}' - \boldsymbol{A}''\boldsymbol{u}) + \boldsymbol{A}^{-1}\sum_{i=1}^{n} N_i(\boldsymbol{G}_i - \boldsymbol{g}_i) + \boldsymbol{A}^{-1}\sum_{j=1}^{m} \boldsymbol{M}_j(H_j - h_j), \tag{5}$$

where the prime implies a derivative with respect to $s$; $N_i$ and $\boldsymbol{M}_j$ are indefinite coefficients which are set so that the equations take a linear and regular form [4]:

$$\boldsymbol{u}'' = \underbrace{k_1(\boldsymbol{g}_1,\ldots,\boldsymbol{g}_n,h_1,\ldots,h_m)}_{\text{const}}\boldsymbol{u} + \underbrace{\boldsymbol{k}_2(\boldsymbol{g}_1,\ldots,\boldsymbol{g}_n,h_1,\ldots,h_m)}_{\text{const}}. \tag{6}$$

For the perturbed case, applying the above transforms, we have weakly nonlinear equations of the form

$$\boldsymbol{u}'' = k_1(\boldsymbol{g}_1,\ldots,\boldsymbol{g}_n,h_1,\ldots,h_m)\boldsymbol{u} + \boldsymbol{k}_2(\boldsymbol{g}_1,\ldots,\boldsymbol{g}_n,h_1,\ldots,h_m) + \boldsymbol{A}^{-1}f^2\boldsymbol{P}. \tag{7}$$

Since here the integral parameters are not constant and, besides, owing to the appearance of $\boldsymbol{P} = \boldsymbol{P}(t)$, the right side becomes a function of time, the system (7) should be completed with the equations

$$\boldsymbol{g}_i' = f\frac{\partial \boldsymbol{G}_i}{\partial \dot{\boldsymbol{x}}}\boldsymbol{P}, \quad h_j' = f\frac{\partial H_j}{\partial \dot{\boldsymbol{x}}}\boldsymbol{P}, \quad t' = f \quad (i=1,\ldots,n;\ j=1,\ldots,m).$$

By a proper choice of transforms (4) and coefficients $N_i$ and $\boldsymbol{M}_j$ one can obtain a multivariate set of systems of differential equations of the form (7). Among these systems of equations are those written in Sperling–Burdet (SB) [5, 6] and Kustaanheimo-Stiefel (KS) variables which are widely used in practice [7]. In the first system, $\boldsymbol{x} = \boldsymbol{u}$ and $dt = |\boldsymbol{x}|\,ds$; the energy integral $h$ and the Laplace integral $\boldsymbol{g}$ are used for the integrals, and the equations have the form

$$\boldsymbol{x}'' - 2h\boldsymbol{x} + \boldsymbol{g} = |\boldsymbol{x}|^2\ \boldsymbol{P}, \quad \boldsymbol{g}' = 2\boldsymbol{x}(\boldsymbol{x}'\cdot\boldsymbol{P}) - \boldsymbol{x}'(\boldsymbol{x}\cdot\boldsymbol{P}) - \boldsymbol{P}(\boldsymbol{x}\cdot\boldsymbol{x}'), \quad h' = (\boldsymbol{x}'\cdot\boldsymbol{P}),$$
$$\tau' = -\frac{1}{2h}\left[\mu + |\boldsymbol{x}|(\boldsymbol{x}\cdot\boldsymbol{P}) - \frac{(\boldsymbol{x}\cdot\boldsymbol{x}')}{|\boldsymbol{x}|}\frac{h'}{h}\right], \quad t = \tau + \frac{1}{2h}\frac{(\boldsymbol{x}\cdot\boldsymbol{x}')}{|\boldsymbol{x}|}. \tag{8}$$

Here, for the determination of the time $t$ a time element $\tau$ is introduced which, for a weakly perturbed motion, behaves almost linearly and, therefore, its integration is more accurate.

In the second system, $\boldsymbol{x} = \boldsymbol{L}(\boldsymbol{u})\boldsymbol{u}$ and $dt = |\boldsymbol{x}|\,ds$, where $\boldsymbol{L}(\boldsymbol{u})$ is the so-called Kustaanheimo-Stiefel matrix [7]. The equations of motion in KS variables can be reduced to a linear and regular form by using only the energy integral, and the system of equations is represented in the form

$$\boldsymbol{u}'' - \frac{h}{2}\boldsymbol{u} = \frac{|\boldsymbol{u}|^2}{2}\boldsymbol{L}^T\boldsymbol{P}, \quad h' = 2(\boldsymbol{u}'\cdot\boldsymbol{L}^T\boldsymbol{P}), \quad \tau' = -\frac{1}{2h}\left[\mu + |\boldsymbol{u}|^2\ (\boldsymbol{u}\cdot\boldsymbol{L}^T\boldsymbol{P}) - 2(\boldsymbol{u}\cdot\boldsymbol{u}')\frac{h'}{h}\right], \quad t = \tau + \frac{1}{h}(\boldsymbol{u}\cdot\boldsymbol{u}'). \tag{9}$$

It should be noted that equations (8) and (9) possess stabilizing properties. Actually, in the nonperturbed case, these equations of motion are reduced to harmonic oscillators, whose solutions are known to be Lyapunov stable. In the perturbed case, the equations of motion are unstable. However, this instability is much less detrimental to the numerical integration than the instability of the classical equations (1).

## 2. SMOOTHING TRANSFORMS

Smoothing transforms are applied to differential equations to reduce the rate of variation of their right sides and thus noticeably increase the accuracy of the numerical integration, especially for strongly elongated orbits.

Smoothing transforms are of the form [8]

$$dt = f(\boldsymbol{x}, \dot{\boldsymbol{x}})ds. \tag{10}$$

Application of the transform (10) implies the change to a new independent variable $s$. After transformation, the equations become

$$\boldsymbol{x}'' = f^2(\boldsymbol{F} + \boldsymbol{P}) + f^{-1}f'\boldsymbol{x}'.$$

For putting the coordinates $\boldsymbol{x}$ into correspondence with the physical time $t$, it is necessary to supplement the system of equations by the addition of the equation $t' = f$. It should be noted that for making the numerical integration more accurate one can, as above, replace the equation of time transformation by the equation in a time element, though the need arises again to extend the system and supplement it with the energy equation.

In the practice of celestial mechanics, smoothing transforms are widely used in which for $f$ the following quantities are chosen: $|\boldsymbol{x}|$ (eccentric anomaly); $|\boldsymbol{x}|^{3/2}$ (elliptic abnormality); $|\boldsymbol{x}|^2$ (true anomaly), and $|\dot{\boldsymbol{x}}|^{-1}$ (an arc of the orbit) [9].

In asteroid problems, where the investigation of an orbit becomes complicated due to an appreciable influence of $N$ principal planets, one can use the following transform:

$$dt = \left(\sum_{i=0}^{N}\frac{M_i}{|\boldsymbol{x} - \boldsymbol{x}_i|}\right)^{-1} ds \equiv \rho ds, \tag{11}$$

where $M_i$ and $\boldsymbol{x}_i$ are, respectively, the mass and coordinates of the *i*th body, where $M_0 = 1$ is the mass of the Sun and $\boldsymbol{x}_0 = \mathbf{0}$. It is of interest that for a close vicinity of the *k*th planet ($|\boldsymbol{x} - \boldsymbol{x}_k| \to 0$) we have $\rho \sim |\boldsymbol{x} - \boldsymbol{x}_k|$; therefore, the transform (10) degenerates into the Sundman transform [10] written for the planet that is treated as a central body.

## 3. NUMERICAL STABILIZATION

The extended system of equations for a perturbed motion (1)

$$\frac{d^2\boldsymbol{x}}{dt^2} = F + P, \quad \frac{dh}{dt} = \frac{\partial H}{\partial \dot{\boldsymbol{x}}} P \tag{12}$$

has the following integral relation:

$$\Delta H \overset{\text{def}}{\equiv} H(\boldsymbol{x}, \dot{\boldsymbol{x}}) - h \equiv 0, \tag{13}$$

which is not fulfilled in performing the numerical integration due to the errors in the integrable variables $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, and $h$, i.e. $\Delta H \neq 0$. Hence, some properties of the motion under investigation, which are expressed by the integral relation (13), are broken.

*3.1. Baumgarte method.* To resolve this problem, Baumgarte [11, 12] suggested to introduce artificial perturbing forces which would lead to asymptotic stability in $\Delta H$ into the equations of motion, so that any error $\Delta H$ would tend to zero in the course of integration. According to Baumgarte, the stabilized equations take the form

$$\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F} + \boldsymbol{P} - \gamma \Delta H \frac{\partial H}{\partial \dot{\boldsymbol{x}}} \left( \frac{\partial H}{\partial \dot{\boldsymbol{x}}} \frac{\partial H}{\partial \dot{\boldsymbol{x}}} \right)^{-1}, \quad \frac{dh}{dt} = \frac{\partial H}{\partial \dot{\boldsymbol{x}}} \boldsymbol{P}, \quad \Delta H = H(\boldsymbol{x}, \dot{\boldsymbol{x}}) - h, \tag{14}$$

where $\gamma$ is the so-called stabilizing parameter, which, generally speaking, is not known before the numerical integration, and it should be chosen experimentally once the best results have been obtained, though it has been shown [13, 14] that when investigating weakly perturbed near-circular orbits the mean motion should be taken for this parameter.

When integrating an orbital motion, it is of great importance that the integral relation for the energy hold (see, e.g., [13]), i.e. that the following condition be satisfied:

$$\Delta H = \frac{\dot{\boldsymbol{x}}^2}{2} - \frac{\mu}{|\boldsymbol{x}|} - h = 0. \tag{15}$$

Based on the formulas of the two-body problem, it is easy to show [13] that if the energies of two close Keplerian solutions differ by $\Delta H$ at some initial point in time $t_0$, this results in a discrepancy between the solutions $\Delta\boldsymbol{x}$ and $\Delta\dot{\boldsymbol{x}}$ according to the linear estimate

$$\begin{pmatrix} \Delta\boldsymbol{x} \\ \Delta\dot{\boldsymbol{x}} \end{pmatrix} = \frac{\Delta n}{n} \begin{pmatrix} \dot{\boldsymbol{x}} \\ \ddot{\boldsymbol{x}} \end{pmatrix} \Delta t, \quad \frac{\Delta n}{n} = \frac{3}{2} \frac{\Delta H}{H}, \tag{16}$$

where $\Delta t = t - t_0$. Actually, relation (16) indicates that the Keplerian motion is Lyapunov unstable. In the course of numerical integration, relation (15) breaks down and the error $\Delta H$ behaves almost linearly with time, whereas the errors $|\Delta\boldsymbol{x}|$ and $|\Delta\dot{\boldsymbol{x}}|$ vary in a quadric (superlinear) fashion according to the estimate (16). The same behavior is observed for the integration of weakly perturbed orbits.

If the energy relation (15) is used for the integral relation (13), the stabilized equations become [12]

$$\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F} + \boldsymbol{P} - \gamma\Delta H \frac{\dot{\boldsymbol{x}}}{|\dot{\boldsymbol{x}}|^2}, \quad \frac{dh}{dt} = \dot{\boldsymbol{x}} \cdot \boldsymbol{P}, \quad \Delta H = \frac{\dot{\boldsymbol{x}}^2}{2} - \frac{\mu}{|\boldsymbol{x}|} - h. \tag{17}$$

*3.2. Nacozy method.* An alternative method of conservation of the integral relations (13) was proposed by Nacozy [15]. Following Nacozy, equations (12) are integrated by a conventional technique; however, after several integration steps the variables $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are corrected for the deviation $H(\boldsymbol{x}, \dot{\boldsymbol{x}})$ from $h$ by the approximate formula

$$\boldsymbol{x}_C = \boldsymbol{x}\left(1 - \frac{\Delta H}{D}\frac{\mu}{|\boldsymbol{x}|^3}\right), \quad \dot{\boldsymbol{x}}_C = \dot{\boldsymbol{x}}\left(1 - \frac{\Delta H}{D}\right), \quad D = \dot{\boldsymbol{x}}^2 + \frac{\mu^2}{|\boldsymbol{x}|^4}, \tag{18}$$

where $\boldsymbol{x}_C$ and $\dot{\boldsymbol{x}}_C$ are the corrected values of the integrable variables. If $H(\boldsymbol{x}_C, \dot{\boldsymbol{x}}_C) \neq h$, the procedure of correction by formula (18) is carried out again for $\boldsymbol{x} = \boldsymbol{x}_C$ and $\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_C$, and so on, until the equality is fulfilled with a given accuracy.

*3.3. Baumgarte conservative method.* If $\Delta n$ in (16) is a variable, $\Delta\boldsymbol{x}$ and $\Delta\dot{\boldsymbol{x}}$ will be estimates by

$$\begin{pmatrix}\Delta\boldsymbol{x}\\ \Delta\dot{\boldsymbol{x}}\end{pmatrix} = \int_{t_0}^{t} \frac{\Delta n}{n}\begin{pmatrix}\dot{\boldsymbol{x}}\\ \ddot{\boldsymbol{x}}\end{pmatrix} dt; \quad \text{therefore,} \quad \frac{d}{dt}\begin{pmatrix}\Delta\boldsymbol{x}\\ \Delta\dot{\boldsymbol{x}}\end{pmatrix} = \frac{\Delta n}{n}\begin{pmatrix}\dot{\boldsymbol{x}}\\ \ddot{\boldsymbol{x}}\end{pmatrix}.$$

Hence, the equations for stabilized solutions $\bar{\boldsymbol{x}}$ and $\dot{\bar{\boldsymbol{x}}}$ can be represented as

$$\frac{d}{dt}\begin{pmatrix}\bar{\boldsymbol{x}}\\ \dot{\bar{\boldsymbol{x}}}\end{pmatrix} = \frac{d}{dt}\begin{pmatrix}\boldsymbol{x}\\ \dot{\boldsymbol{x}}\end{pmatrix} - \frac{d}{dt}\begin{pmatrix}\Delta\boldsymbol{x}\\ \Delta\dot{\boldsymbol{x}}\end{pmatrix} = \frac{\bar{n}}{n}\begin{pmatrix}\dot{\boldsymbol{x}}\\ \ddot{\boldsymbol{x}}\end{pmatrix} = \left(\frac{h}{H}\right)^{3/2}\begin{pmatrix}\dot{\boldsymbol{x}}\\ \ddot{\boldsymbol{x}}\end{pmatrix}, \tag{19}$$

where $\bar{n} = n - \Delta n$ is the mean motion of the stabilized orbit. Substituting the stabilized solutions in the right side of (19), we obtain the equations

$$\frac{d}{dt}\begin{pmatrix}\bar{\boldsymbol{x}}\\ \dot{\bar{\boldsymbol{x}}}\end{pmatrix} = \left(\frac{h}{H}\right)^{3/2}\begin{pmatrix}\dot{\bar{\boldsymbol{x}}}\\ \ddot{\bar{\boldsymbol{x}}}\end{pmatrix}, \quad \ddot{\bar{\boldsymbol{x}}} = -\frac{\mu}{|\bar{\boldsymbol{x}}|^3}\bar{\boldsymbol{x}} + \boldsymbol{P}(t, \bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}}). \tag{20}$$

Equations (20) were first derived by Baumgarte [8, 16], though in a different way. This method of stabilization, in contrast to the previous ones, gives no change in energy; therefore, the author has called it conservative. Unfortunately, the conservative method has appeared inefficient as apllied to problems of celestial mechanics. This is related in the main to the multiplier $(h/H)^{3/2}$ present in equations (20) that complicates the behavior of the right sides.

*3.4. Stabilization in time.* The undesirable multiplier in (20) can be eliminated by integrating along a new time $\bar{t}$ which is related to $t$ as

$$dt = d\bar{t}\left(\frac{H}{h}\right)^{3/2}. \tag{21}$$

As a result, the new equations become

$$\frac{d}{d\bar{t}}\begin{pmatrix}\bar{\boldsymbol{x}}\\ \dot{\bar{\boldsymbol{x}}}\end{pmatrix} = \begin{pmatrix}\dot{\bar{\boldsymbol{x}}}\\ \ddot{\bar{\boldsymbol{x}}}\end{pmatrix}, \quad \frac{dh}{d\bar{t}} = (\dot{\bar{\boldsymbol{x}}} \cdot \boldsymbol{P})\left(\frac{H}{h}\right)^{3/2}, \quad \frac{dt}{d\bar{t}} = \left(\frac{H}{h}\right)^{3/2}, \quad H = \frac{\dot{\bar{\boldsymbol{x}}}^2}{2} - \frac{\mu}{|\bar{\boldsymbol{x}}|}. \tag{22}$$

This stabilization method has the disadvantage that the system of equations must include an additional equation in $t$. Moreover, since $t$ is treated as an integrable variable, a problem is to come at a given point in true time. This problem can be solved as follows. Integrating numerically up to $\bar{t} = t$, we obtain $t = t(\bar{t})$. If the value of $t - \bar{t}$ is small enough, we

seek a solution by the approximate formula $\bar{\boldsymbol{x}}(t) = \bar{\boldsymbol{x}}(\bar{t}) + \dot{\bar{\boldsymbol{x}}}(\bar{t})(t-\bar{t}) + \ddot{\bar{\boldsymbol{x}}}(\bar{t})(t-\bar{t})^2/2$. If $t-\bar{t}$ is great, we again perform integration up to $\bar{t} = t$, and so on, until $t-\bar{t}$ becomes negligible.

## 4. METHOD OF VARIATION OF ARBITRARY CONSTANTS

The method of variation of arbitrary constants, first proposed by Newton and developed in detail by Lagrange, was originally used in general perturbation theory, but it is also applied successfully in performing numerical integration.

In the method of variation of constants, an orbit is represented in nonperturbed form and the orbit parameters (elements) are treated as variables. According to the form of representation of the orbit, differential equations in the orbit parameters are constructed which then are integrated by approximate methods. It should be noted that the Lagrangian technique can be applied not only to constant parameters, but also to various variable functions of these parameters and time.

In the case of smooth perturbations, the parameters are slow variables; therefore, their equations will be integrated by numerical methods much more efficiently than the classical equations.

From the computational viewpoint, the equations in Keplerian elements are inconvenient in that they are rather complicated, contain many trigonometric functions, and, moreover, have singularities for zero declinations and eccentricities. To circumvent these difficulties, Roy suggested to treat the angular momentum $\boldsymbol{c}$, the Laplace vector $\boldsymbol{g}$, and the true longitude $\lambda = v + \Omega + \omega$ as orbit parameters. Their equations can be written in the form [2]

$$\frac{d\boldsymbol{c}}{dt} = \boldsymbol{x} \times \boldsymbol{P}, \quad \frac{d\boldsymbol{g}}{dt} = \boldsymbol{P} \times \boldsymbol{c} + \dot{\boldsymbol{x}} \times \dot{\boldsymbol{c}}, \quad \frac{d\lambda}{dt} = \frac{|\boldsymbol{c}|}{|\boldsymbol{x}|^2} + \frac{c_1\dot{c}_2 - c_2\dot{c}_1}{|\boldsymbol{c}|(|\boldsymbol{c}| + c_3)}. \tag{23}$$

The system of equations (23) has a serious disadvantage, which shows up in the main in the integration of strongly elongated orbits. The last equation in (23) is singular at the origin of coordinates, and this results in an excentric behavior of its right side. To circumvent this difficulty, it suffices to replace $\lambda$ by the mean longitude $l = M + \Omega + \omega$, which varies almost linearly in a weakly perturbed motion.

## 5. THE ENCKE METHOD

Encke's idea [1, 2, 17] is to integrate, instead of the coordinates, their perturbations (deviations) relative to a previously chosen (as a rule, Keplerian) orbit. The use of the Encke method will be efficient only in the event that the perturbations to be integrated are not only less, but also smoother than the orbit, since this reduces the methodical error and, hence, the integration step. The property of smoothness is primary in the Encke method, and the perturbations should not be necessarily small. Nevertheless, in practice, their smallness is an obligatory requirement since this allows one to reduce the effect of round-off errors on the representation of the orbit to be computed.

In the classical Encke method, for the reference orbit an osculating (at the initial point in time) Keplerian orbit $\boldsymbol{x}_\mathrm{K} = \boldsymbol{x}_\mathrm{K}(t)$ is chosen which is described by the equations

$$\frac{d^2\boldsymbol{x}_\mathrm{K}}{dt^2} = \boldsymbol{F}(\boldsymbol{x}_\mathrm{K}). \tag{24}$$

Then the differential equations in perturbations $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_\mathrm{K}$ become

$$\frac{d^2\delta\boldsymbol{x}}{dt^2} = \boldsymbol{F}(\boldsymbol{x}_\mathrm{K} + \delta\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x}_\mathrm{K}) + \boldsymbol{P}(t, \boldsymbol{x}_\mathrm{K} + \delta\boldsymbol{x}, \dot{\boldsymbol{x}}_\mathrm{K} + \delta\dot{\boldsymbol{x}}). \tag{25}$$

It should be noted that for small $\delta\boldsymbol{x}$ the Keplerian terms in (25) are close in value. At the same time, their small difference is much less than each of them, and, hence, it will be computed with an unsatisfactory accuracy because of round-off errors. To resolve this problem, additional transformations are performed which reduce the difference of Keplerian terms to a certain function **Φ,** such that its smallness is explicitly expressed in terms of the small perturbations $\delta\boldsymbol{x}$ , and the equations take the final form

$$\frac{d^2\delta\boldsymbol{x}}{dt^2} = \boldsymbol{\Phi}(\delta\boldsymbol{x}) + \boldsymbol{P}(t, \boldsymbol{x}_\mathrm{K} + \delta\boldsymbol{x}, \dot{\boldsymbol{x}}_\mathrm{K} + \delta\dot{\boldsymbol{x}}). \tag{26}$$

In equations (26), the function **Φ** can be reduced, for example, to the following form [18]:

$$\boldsymbol{\Phi}(\delta\boldsymbol{x}) = -\frac{\mu}{|\boldsymbol{x}|^3}\boldsymbol{x} + \frac{\mu}{|\boldsymbol{x}_\mathrm{K}|^3}\boldsymbol{x}_\mathrm{K} = \frac{\mu}{|\boldsymbol{x}_\mathrm{K}|^3}(D\boldsymbol{x} - \delta\boldsymbol{x}), \quad D = \left(1 + \frac{\zeta^2}{1+\zeta}\right)\frac{(\boldsymbol{x}_\mathrm{K} + \boldsymbol{x})\delta\boldsymbol{x}}{|\boldsymbol{x}|^2}, \quad \zeta = \frac{|\boldsymbol{x}_\mathrm{K}|}{|\boldsymbol{x}|}.$$

Even if the perturbations $\delta\boldsymbol{x}$ are small in the beginning of integration, they increase with time. On rather long time intervals they increase to an extent that the function **Φ** (Encke term) becomes comparable to the perturbing forces $\boldsymbol{P}$ . In this case, the orbit is corrected by re-calculating the parameters of the reference motion for the new epoch.

It should be noted that the Encke method is not restricted to equations in rectangular coordinates (1). It can be also applied successfully, for example, to equations (8) [19] or (9) [20]. Thus, if we take for the reference solution the nonperturbed solution of (9),

$$\boldsymbol{u}_\mathrm{K} = \boldsymbol{u}_0 \cos\omega s + \frac{\boldsymbol{u}_0'}{\omega}\sin\omega s, \quad \omega = \sqrt{-\frac{h_\mathrm{K}}{2}}, \quad h_\mathrm{K} = h_0, \quad \tau_\mathrm{K} = -\frac{1}{h_\mathrm{K}}(\boldsymbol{u}_0 \cdot \boldsymbol{u}_0') - \frac{\mu}{2h_\mathrm{K}}s,$$

the Encke equations in the KS interpretation will have the form [20]

$$\delta\boldsymbol{u}'' - \frac{h_\mathrm{K}}{2}\delta\boldsymbol{u} - \frac{\delta h}{2}\boldsymbol{u} = \frac{|\boldsymbol{u}|^2}{2}\boldsymbol{L}^T\boldsymbol{P}, \quad \delta h' = 2(\boldsymbol{u}' \cdot \boldsymbol{L}^T\boldsymbol{P}), \quad \delta\tau' = -\frac{1}{2h}\left[-\frac{\delta h}{h_\mathrm{K}}\mu + |\boldsymbol{u}|^2\,(\boldsymbol{u} \cdot \boldsymbol{L}^T\boldsymbol{P}) - 2(\boldsymbol{u} \cdot \boldsymbol{u}')\frac{h'}{h}\right], \tag{27}$$

where the perturbed quantities are determined as the sum of their reference analogs and corresponding perturbations.

## 6. THE PROBLEM OF SHORT-PERIOD PERTURBATIONS

As we already noted, the use of the Encke method is worthwhile if the perturbations are smoother than the orbit. The "roughness" of an orbit is caused, as a rule, by short-period perturbing forces [21]. With the restricted circular three-body problem as an example, it can be decided whether the Encke method is applicable to the particular case of short-period perturbations.

In this problem, the orbit of the third body is perturbed by an internal massive body which rapidly rotates around the central body. The equations of the problem can be represented in the form

$$\frac{d^2\boldsymbol{x}}{dt^2} = \underbrace{-\mu\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}}_{\boldsymbol{F}} \underbrace{-\mu_P\frac{\boldsymbol{x} - \boldsymbol{x}_P}{|\boldsymbol{x} - \boldsymbol{x}_P|^3} - \mu_P\frac{\boldsymbol{x}_P}{|\boldsymbol{x}_P|^3}}_{\boldsymbol{P}}, \tag{28}$$

where $\mu_P$ and $\boldsymbol{x}_P$ are the gravitational parameter and the position vector of the second body, respectively, and $\mu_P \ll \mu$ . Note that these equations are written in the coordinates related to the central body.

The Encke method actually eliminates the effect of the Keplerian term $\boldsymbol{F}$ and leaves the integration of the perturbing forces $\boldsymbol{P}$ to a numerical method. Hence, the Encke method is efficient only when the function $\boldsymbol{P}$ is smoother than $\boldsymbol{F}$ or, in other words, when the perturbations determined by the force $\boldsymbol{P}$ are integrated more rapidly than the Keplerian orbit.

If a numerical method of order $p$ is used, for the error of integration of a near-circular Keplerian orbit $|\Delta \boldsymbol{x}_{\mathrm{K}}|$ at a step $\Delta t$ the following estimate is valid:

$$|\Delta \boldsymbol{x}_{\mathrm{K}}| \approx a(n\Delta t)^{p+1}/(p+1)! \tag{29}$$

The inner massive body causing short-period perturbations in the orbit of the third one should be close enough to the central body. Then the second term (inertial force) will dominate in $\boldsymbol{P}$. The error of integration at the step $\Delta t_P$ of this part can be estimated as

$$|\Delta \boldsymbol{x}_P| \approx \beta a_P (n_P \Delta t_P)^{p+1}/(p+1)!, \tag{30}$$

where $\beta = \mu_P/(\mu+\mu_P)$; $a$ and $a_P$ are the major semiaxes of the third and the second body, respectively; and $n$ and $n_P$ are their mean motions. Let us introduce a small parameter $\alpha = a_P/a$, such that $\alpha^{-3/2} = n_P/n$. Then, according to (29) and (30), the same integration accuracy is achievable for $\boldsymbol{F}$ and $\boldsymbol{P}$ only if the steps $\Delta t$ and $\Delta t_P$ satisfy the relation

$$\nu \equiv \Delta t/\Delta t_P = (\alpha\beta)^{1/(p+1)}\alpha^{-3/2}. \tag{31}$$

Thus, if $\Delta t_P > \Delta t$ or $\nu < 1$, the use of the Encke method is expedient, which otherwise would not be so. Moreover, the smaller the value of $\nu$, the more efficient is the Encke method. This estimate can also serve to justify the use of the other methods considered.

It should be noted that the integration of circular orbits is performed with a constant step; therefore, the coefficient (31) can be calculated as $\nu = M_P/M$, where $M$ and $M_P$ are the numbers of steps $\Delta t$ and $\Delta t_P$ fulfilled throughout the integration interval.

For the elliptic case, we have noticed that to provide a certain local error, the integration step should be chosen to fit a constant step in the elliptic anomaly. In this case, the number of steps per turn, $M_e$, increases with orbit eccentricity as

$$M_e = M\sigma, \quad \sigma = \frac{1}{\sqrt{1-e}}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-\varepsilon\sin^2\varphi}}, \quad \varepsilon = -\frac{2e}{1-e}. \tag{32}$$

In order that the integration accuracy correspond to that in the circular case, the step in the elliptic anomaly should be reduced by multiplying it by $\xi^{1/(p+1)}$, where $\xi = \sqrt{1-e}$. As a result, we obtain the generalized coefficient

$$\nu \equiv \frac{1}{\sigma}(\alpha\beta\xi)^{1/(p+1)}\alpha^{-3/2}. \tag{33}$$

However, the inertia term can be eliminated by writing the equations of motion with respect to the barycenter of the massive bodies. These equations can be represented in the form of (1):

$$\frac{d^2\boldsymbol{x}_B}{dt^2} = \underbrace{-(\mu+\mu_P)\frac{\boldsymbol{x}_B}{|\boldsymbol{x}_B|^3}}_{\boldsymbol{F}} \underbrace{\overbrace{-\mu\frac{\boldsymbol{x}_B-\boldsymbol{x}_{1B}}{|\boldsymbol{x}_B-\boldsymbol{x}_{1B}|^3}+\mu\frac{\boldsymbol{x}_B}{|\boldsymbol{x}_B|^3}}^{\boldsymbol{P}_1} \overbrace{-\mu_P\frac{\boldsymbol{x}_B-\boldsymbol{x}_{2B}}{|\boldsymbol{x}_B-\boldsymbol{x}_{2B}|^3}+\mu_P\frac{\boldsymbol{x}_B}{|\boldsymbol{x}_B|^3}}^{\boldsymbol{P}_2}}_{\boldsymbol{P}}, \tag{34}$$

where $\boldsymbol{x}_{1B}$, $\boldsymbol{x}_{2B}$, and $\boldsymbol{x}_B$ are the barycentric position vectors of the first, the second, and the third body, respectively. Here $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ can be treated as the perturbing force responsible for the deviation of the massive bodies from the barycenter. It should be noted that $\boldsymbol{x}_{1B}$ and $\boldsymbol{x}_{2B}$ are small; therefore, the Encke transform should be applied to $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. Besides, from the condition $\mu_P \ll \mu$ it follows that $\boldsymbol{P}_1 \ll \boldsymbol{P}_2$ (see, e.g., [21]), and, hence, in our case the perturbing force $\boldsymbol{P}_1$ can be neglected.

It is generally rather problematic to estimate $|\Delta \boldsymbol{x}_P|$ for $\boldsymbol{P}_2$. However, we have been able to obtain its limiting values for the circular plane problem with the use of a numerical method:

$$|\Delta \boldsymbol{x}_P|_{\min}^{\max} \approx \pm \frac{a(n\Delta t_P)^{p+1}}{(p+1)!} \frac{\beta}{(1 \mp \alpha)^{p+1}} \sum_{i=0}^{p-1} (1 \mp \alpha)^i (\alpha^{-3/2} - 1)^{p-1-i} \sum_{j=1}^{p-1-i} a_{ij} (\pm\alpha)^j . \tag{35}$$

Here $a_{ij}$ are integers depending on $p$.

Assume that for some $\alpha_B$ the error $|\Delta \boldsymbol{x}_P|$ (30) is equal to $|\Delta \boldsymbol{x}_P|^{\max}$ (35). Then for all $\alpha > \alpha_B$ the transition to a barycentric system becomes unreasonable. Empirically, we have obtained the following approximate formula for the boundary value of $\alpha_B$: $\alpha_B = e^{-0.2p}$.

For relatively small $\alpha$ we have

$$|\Delta \boldsymbol{x}_P| \approx |\Delta \boldsymbol{x}_P|^{\max} \approx |\Delta \boldsymbol{x}_P|_{\min} \approx 2a(n\Delta t_P)^{p+1} \beta\alpha(\alpha^{-3/2})^{p-1}/(p+1)!,$$

whence the coefficient $\nu_B$ for equation (34) will be

$$\nu_B = (2\beta\alpha)^{1/(p+1)} (\alpha^{-3/2})^{(p-1)/(p+1)}. \tag{36}$$

Therefore, comparing (31) and (36), one can expect that on going to a barycentric system the speed of numerical integration will increase by a factor of $(2\alpha^3)^{-1/(p+1)}$.

However, notwithstanding that equations (34) will be integrated more rapidly than (28), in practice this approach, as a rule, fails to solve the problem of short-period perturbations, in particular when using high-order numerical methods, and the application of any methods of special perturbation theory to (34) will also be unsuccessful.

On the other hand, it is of interest that the change to a barycentric coordinate system can be considered an independent method for increasing the efficiency of the numerical integration; however, it makes sense to use this method only if $\alpha$ are small enough.

Thus, it is our belief that the methods of special perturbation theory will be efficient only as applied to the problems where short-period perturbing forces are absent, or, if they are present, only if $\nu \ll 1$. For example, these methods are well applicable to the problems of the dynamics of close satellites or terrestrial planets, and this will be demonstrated in the next work.

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