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SPECIAL PERTURBATION THEORY METHODS IN CELESTIAL MECHANICS. II. COMPARATIVE ANALYSIS OF NUMERICAL EFFICIENCY

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A comparative analysis of the efficiency of methods in special perturbation theory [1] is performed as applied to the numerical simulation of satellite, asteroid, and planetary orbits, and recommendations on their use are given.

INTRODUCTION

In the first part of the present work [1] the basic ideas and principles for the construction of methods in special perturbation theory are proposed and their application to the solution of problems in celestial mechanics is substantiated. To give clear recommendations on using the methods, a comparative analysis of their efficiency as applied to numerical simulation of satellite, asteroid, and planetary orbits is performed.

1. NUMERICAL EXPERIMENT

The numerical efficiency of methods in special perturbation theory (Table 1) was investigated as applied to the solution of problems on the dynamics of satellites, asteroids, and planets (Table 2). Table 1 gives references to earlier derived formulas [1]; the column G presents designations of the corresponding integration variables for the characteristics of the numerical efficiency of methods in the figures given below; N is the number of equations to be integrated. It should be noted that in the Encke methods [1] the reference solutions were recalculated after every two turns. In Table 2, T, e, and i are, respectively, the period, eccentricity, and inclination of the orbit under investigation (the inclinations are given relative to the equator of the central planet for satellites and relative to the ecliptic for asteroids); J is the coefficient of the second zonal harmonic of the planet, and ν is the coefficient of influence of short-period perturbations [1]. For the problems under consideration the following perturbing factors were considered: the effect of nonsphericity of the central body (for satellites), J; the attraction by massive bodies (for satellites and planets), ◐; and the attraction by the Sun (for satellites), ☼. Other perturbing factors were not considered in view of their insignificant effect on the efficiency of numerical simulation.

1.1. Objects. The satellite motion is considered for three natural satellites of Jupiter and Mars (Amalthea, Himalia, and Phobos) and two artificial Earth satellites (low-flying at an altitude of 300 km and geosynchronous). The dynamics of natural satellites is considered on a 100-year interval: approximately so much time has passed from the time of their discovery.

Amalthea and Phobos are close moons, moving near the plane of the planet equator in near-circular orbits. Their orbits are strongly perturbed due to the nonsphericity of the central planet. In addition, Jupiter's moon experiences a strong gravitational influence of massive Galilean moons. The orbital periods of the objects are less than one day, and they make a rather large number of turns – more than 50 000 – for a 100-year interval. Therefore, their dynamics should be considered long-term from the viewpoint of numerical integration.

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TABLE 1. Special Perturbation Theory Methods

Variables	G	Reference	N	Variables	G	Reference	N
$\boldsymbol{x}, \dot{\boldsymbol{x}}$ (t)	**x**	(1)	6	$\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{g}, h, \tau$ (s)	**sb**	(8)	11
$\delta\boldsymbol{x}, \delta\dot{\boldsymbol{x}}$ (t)	**δx**	(26)	6	$\boldsymbol{u}, \boldsymbol{u}', h, \tau$ (s)	**u**	(9)	10
$\boldsymbol{x}, \dot{\boldsymbol{x}}, h$ (t)	**st**	(17)	7	$\delta\boldsymbol{u}, \delta\boldsymbol{u}', \delta h, \delta\tau$ (s)	**δu**	(27)	10
$\boldsymbol{x}, \dot{\boldsymbol{x}}, h$ (t)	**nz**	(18)	7	$\boldsymbol{x}, \dot{\boldsymbol{x}}, t$ (s)	**sm**	(10)	7
$\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}}, h$ (t)	**cn**	(20)	7	$\boldsymbol{x}, \dot{\boldsymbol{x}}, t$ (E_G)		(11)	7
$\bar{\boldsymbol{x}}, \dot{\bar{\boldsymbol{x}}}, h, t$ $(\bar{t})$	**δt**	(22)	8	$\boldsymbol{x}, \dot{\boldsymbol{x}}, h, \tau$ (E)	**τ**	(8)	8
$\boldsymbol{c}, \boldsymbol{g}, l$ (t)	**ry**	(23)	7	$\boldsymbol{x}_B, \dot{\boldsymbol{x}}_B$ (t)	**br**	(34)	6

TABLE 2. Celestial Objects and Their Orbits

Object	Center	T, day	e	i, °	Interval, turns/year		Perturbations			
Satellites										
Amalthea	Jupiter	0.499	0.003	0.3	100	73000	J	◐	☼	$J = 3.4 \cdot 10^{-3}$
Himalia	Jupiter	247.767	0.166	30.2	100	152		◐	☼	$\nu = 41$ (Io)
Phobos	Mars	0.319	0.015	1.1	100	114500	J		☼	$J = 3.8 \cdot 10^{-4}$
300 km	Earth	0.063	0.000	50.0	0.038	222	J	◐	☼	$J = 1.5 \cdot 10^{-3}$
GSS	Earth	0.997	0.010	10.0	40	14600	J	◐	☼	$J = 3.7 \cdot 10^{-5}$
Asteroids and planets										
Phaeton	Sun	523.609	0.890	22.2	1433	1000		◐		$\nu = 0.9$ (Mercury)
1 Ceres	Sun	1680.907	0.079	10.6	4602	1000		◐		$\nu = 3.9$ (Mercury)
Mercury	Sun	87.969	0.205	7.0	240	1000		◐		
Jupiter	Sun	4339.289	0.048	1.3	11880	1000		◐		$\nu = 9.4$ (Mercury)

The far moon, Himalia, has a long orbital period and makes only 152 turns for 100 years. The main perturbing factor is its attraction by the Sun; at the same time, the integration of the orbit of Himalia is substantially complicated due to the short-period perturbations from the Galilean moons (mainly, from Io), retarding the numerical process.

The low-flying artificial Earth satellite (300 km altitude) is a fast circumterrestrial object. It makes 222 turns for only two weeks. This is precisely the interval on which we considered its orbital motion. First, we were interested whether the methods under study are appropriate for use on such a short time interval as applied to the integration of a rather smooth orbit; second, it makes no sense to consider the motion of low satellites on longer time intervals, since such a satellite is often subject to orbital correction for the reduction of its altitude due to the atmosphere drag. We, however, did not consider the atmosphere drag because of its insignificant effect on the efficiency of numerical integration.

The geosynchronous object was of interest to us only as a representative of the most polluted area of the circumterrestrial space, being one of numerous fragments of space dust. Its dynamics was simulated on the 40-year time interval which was in fact comparable to the era of space exploration (and pollution).

In the asteroid and planetary problems, the motion of objects was simulated on the interval of 1000 turns. We have considered four objects: two asteroids, Phaeton and Ceres, and two planets, Mercury and Jupiter. Phaeton has a highly elongated orbit with a complex structure of perturbations. We have taken this object as an example to demonstrate the capabilities of regularizing transform. The orbits of Ceres and Jupiter are almost circular. However, their integration is complicated by the short-period perturbations from terrestrial planets, in particular from Mercury. In simulating the motion of Mercury, the short-period perturbation problem does not arise, and this will also be shown below.

1.2. The Everhart method. To integrate the equations of motion, we used the Everhart method [2], widely known among celestial mechanics experts, that was developed by the author specially for numerical simulation of orbits. The Everhart method is a Runge–Kutta–Batcher-type implicit single-step method [3] based on a polynomial approximation of the solution. With this method, to increase the approximation order, the Gauss–Rado partition is used. Thus, for the *k*th-order partition, the method has an order of $2k-1$. We have chosen the method of order 15.

However, it should be noted that with the integration parameters recommended by the author[1] the error of the method not always corresponds to the chosen method order.[1] For example, we have come up with the fact that when the 15th-order Everhart method is used to integrate near-circular orbits (in rectangular coordinates), the global integration error $|\Delta \boldsymbol{x}|$ depending on the step Δt behaves as $|\Delta \boldsymbol{x}| : \Delta t^{10}$, i.e. as in the method of order 10. Nevertheless, with the recommended parameters, the required accuracy of the solution is attained with the least computer time [2].

The method involves an algorithm for choosing a variable integration step. The step is chosen so that the value of the polynomial term of order k be not over 10^{-L} [2], where the parameter L is set by the user. In this connection, the step choosing algorithm proposed by the author of the method is not optimum since the step can appear much less than its admissible value corresponding to the method of order $2k-1$.

1.3. Characteristics of the numerical integration efficiency. To estimate the integration accuracy, we varied the parameter L and estimated the global error of the solution with a smaller L by the solution with a greater L. The numerical integration speed was estimated by the number of steps. As a result, for each system of integrated equations we obtained relations between integration accuracy and integration speed which were used to analyze the efficiency of methods in theory of special perturbations.

It should be noted that the estimation of the integration speed by the number of integration steps is well substantiated. Certainly, the proposed methods complicate the equations of orbital motion, and this, naturally, affects the time of calculation of their right sides. Therefore, it seams that each integration step should be differently labor-consuming for different systems. Nevertheless, it should be remembered that in the modern models the most time-consuming the calculation of the perturbing function $\boldsymbol{P}$, which is invariably present in all systems considered by us, and all artefacts appearing in the equations after transformations merely fade on the background of the conglomerate $\boldsymbol{P}$. Therefore, to estimate the integration speed in our analysis of the method efficiency it suffices to know only the number of references to $\boldsymbol{P}$ or the number of steps proportional to $\boldsymbol{P}$ that have been passed throughout the integration process.

Besides, this characteristic of integration speed is remarkable for its independence of either the optimization of the numerical model or the capabilities of the computer processor. Therefore, its use eliminates the factors that affect the speed of numerical integration, but, at the same time, have no relation to our methods.

2. NUMERICAL RESULTS. SATELLITE PROBLEMS

The accuracy-speed characteristics are given in Figs. 1–11. For the estimation of the significance of integration errors, the dashed lines indicate two levels: one corresponds to the semimajor axis of the orbit (*a*) and the other to an about one-second angular error relative to the terrestrial observer ($1''$).

The results obtained for the satellite problems suggest that the methods of special perturbation theory are most efficient and can be recommended for use only for the long-term numerical simulation of satellite orbits with a smooth structure of perturbations (Figs. 1 and 4*b*). Impressing results are obtained when the KS equations (**u**, **δu**) and the Roy equations (**ry**) are used. Thus, using this models makes it possible to increase the integration speed 3–7 times with the integration accuracy preserved. Moreover, in the case of Phobos, the Encke method with KS variables (**δu**) increases the highest possible accuracy almost by two orders of magnitude due to the lower effect of round-off errors.

These equations are highly efficient, first of all, because they possess a stabilizing effect. Even so, the use of the stabilized equations (**st**) is much less efficient. This, perhaps, is due to that these equations contain artificially introduced

[1] Since the method is implicit, the solution on a step is found by iterations; the initial approximate values for the coefficients of the approximating polynomial on a running step are estimated by the data on the solution for the previous step. In view of that these initial approximate values are rather close to the true values of the coefficients, Everhart recommends to execute only two iterations per step, though the iterative process sometimes yields rather rough values of the coefficients, especially, if the step is rather large. Therefore, the local accuracy appears rather low and does not correspond to the order of the method.

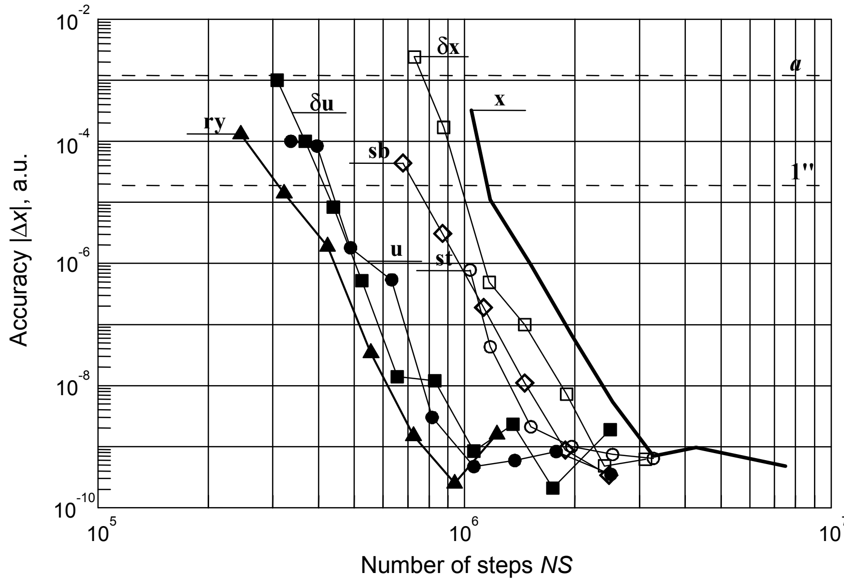

Fig. 1. Amalthea.

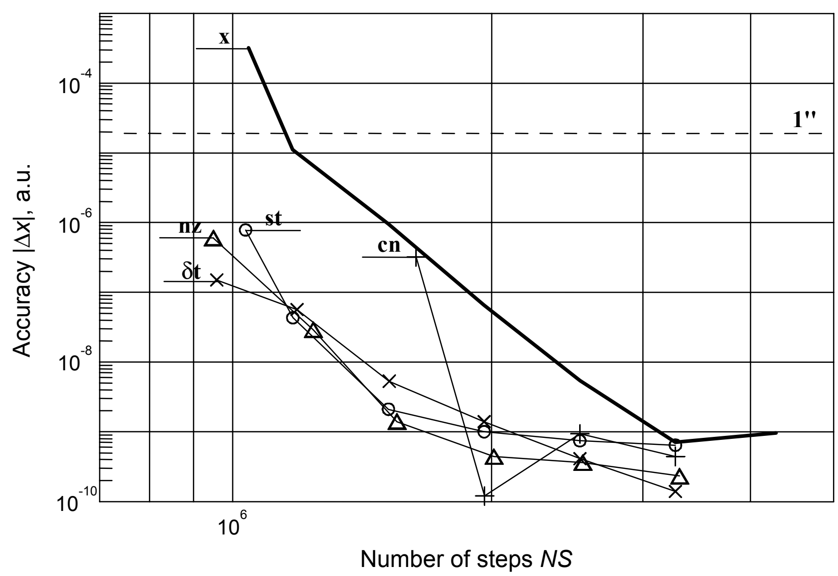

Fig. 2. Amalthea (stabilizing transforms).

terms (stabilizing perturbations) whose behavior is associated with integration errors having no relation to the physics of the problem. Therefore, the given artefact, although well substantiated from the viewpoint of stabilization as a means for Lyapunov instability control, has a side effect that, in the course of time, noticeably distorts the dynamic pattern of the problem.

Using Amalthea as an example (Fig. 2), we have estimated the efficiency of various stabilizing transforms (**st**, **nz**, **cn**, **tδ**) [1]. It turned out that all stabilizing approaches, except for the canonical one (**cn**), are equally good. The low efficiency of the canonical stabilization is accounted for in the first part of the present work [1].

Next it has been shown (Fig. 3) that when simulating the motion of a distant satellite (Himalia), all attempts to increase the numerical integration efficiency by using the considered methods fail because of the short-period perturbations

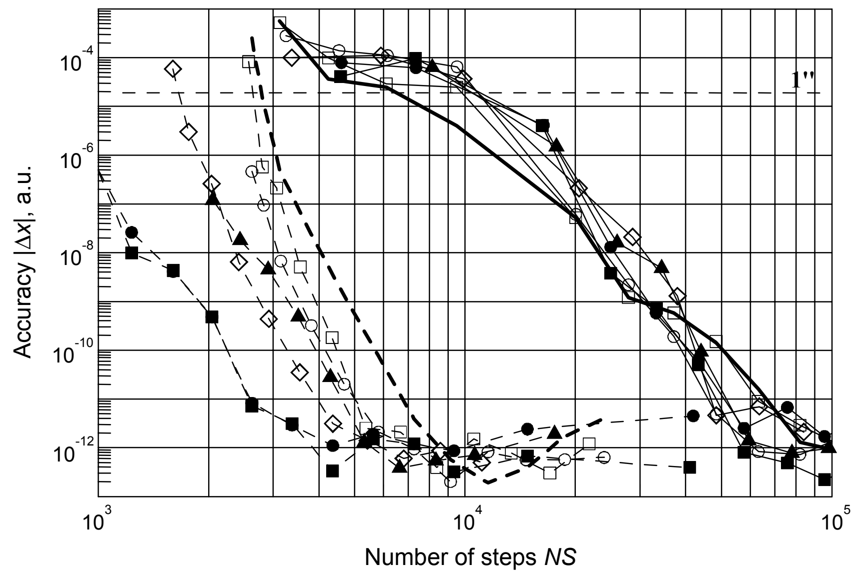

Fig. 3. Himalia.

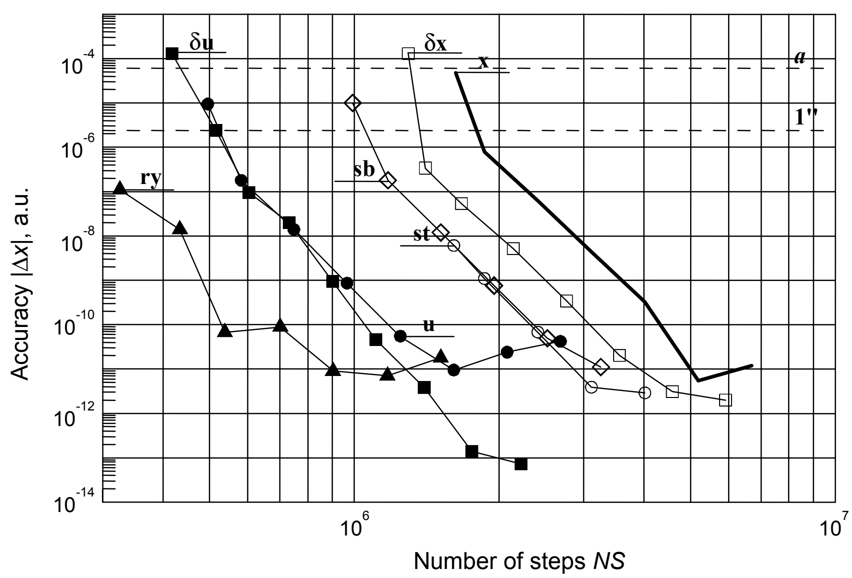

Fig. 4. Phobos.

from the Galilean moons [1]. From Fig. 3 it can also be seen that the integration is much more efficient in the absence of short-period perturbations (dashed lines), and the characteristics are distributed almost in the same order of significance as for the close moons.

Finally, it should be noted that the efficiency of the methods of special perturbation theory noticeably decreases with decreasing time interval (Fig. 5). However, even so, they still remain applicable.

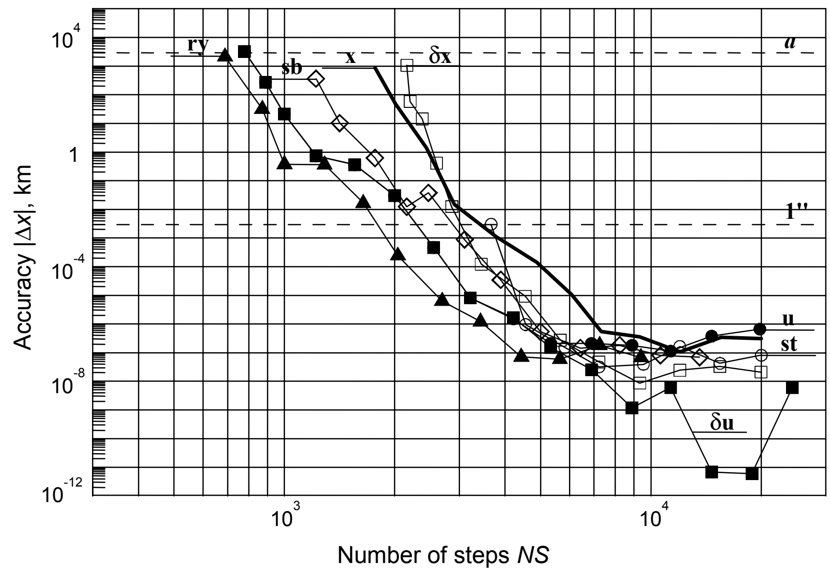

Fig. 5. The 300-km artificial satellite.

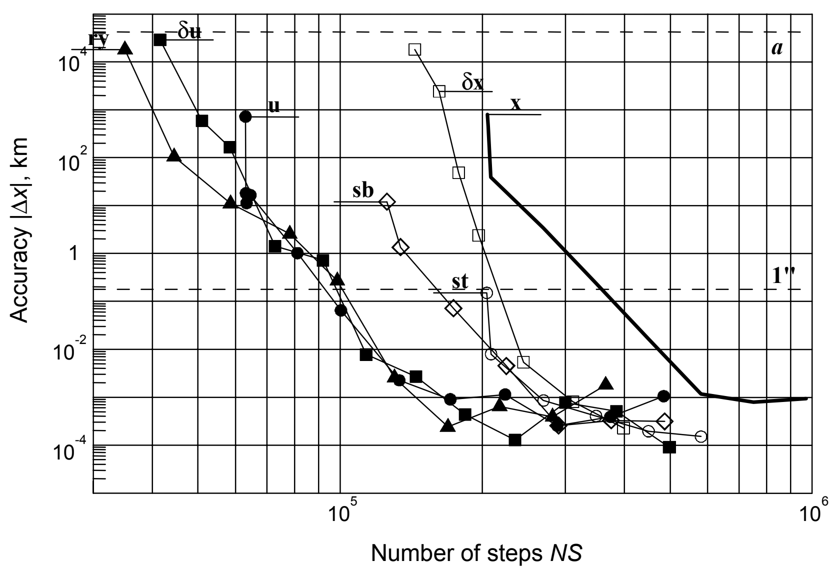

Fig. 6. The geosynchronous artificial satellite.

3. NUMERICAL RESULTS. ASTEROID AND PLANETARY PROBLEMS

The quality of the results obtained for the asteroid and planetary problems depends in the main on the magnitude of the short-period perturbations resulting, mainly, from the gravitational influence of Mercury.

For the asteroid Phaeton, the factor of significance of short-period perturbations, ν [1], is less than unity. Therefore, the use of the methods of special perturbation theory is well substantiated, and this is confirmed by the characteristics given in Fig. 7. In this case, the highest numerical integration efficiency is achieved by using regularizing transforms (**u**, **δu, sb**), and this is an expected result since Phaeton has a very elongated orbit with eccentricity $e = 0.89$.

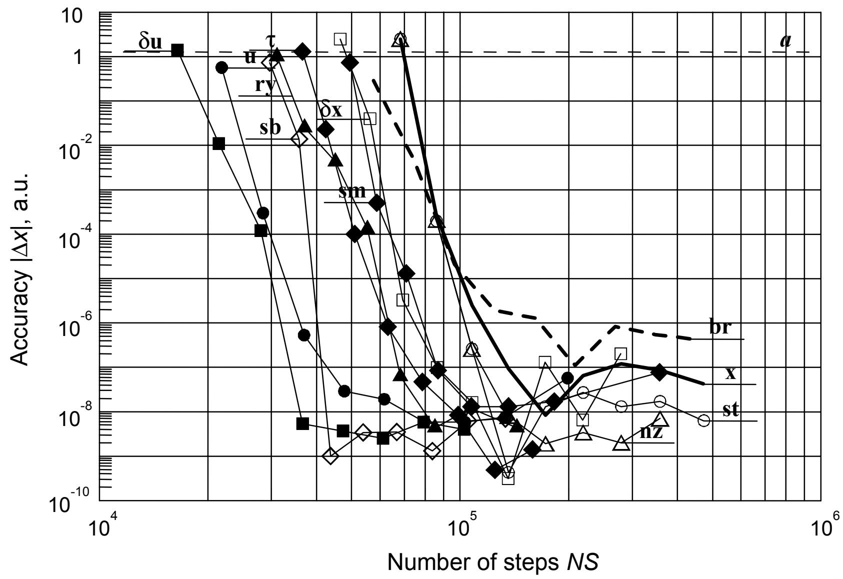

Fig. 7. Phaeton.

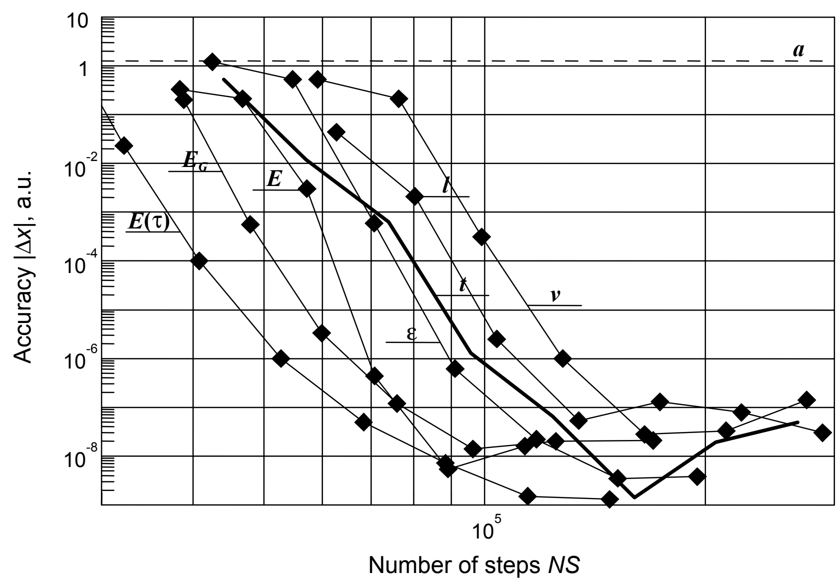

Fig. 8. Phaeton (smoothing transforms).

We also investigated the capabilities of smoothing transforms (**sm**) by the example of Phaeton. The results are presented in Fig. 8. Here, the characteristics correspond to the following independent variables: t is the time; E, ε, v, and E_G are the eccentric, the elliptic, the true, and the generalized eccentric anomaly, respectively; l is an arch of the orbit, and $E(\tau)$ is the eccentric anomaly (with the integration variable τ). As can be seen from the figure, to integrate strongly eccentric orbits with a complex structure of perturbations, as that of Phaeton, it is appropriate to use smoothing transforms where the eccentric anomaly and its analogs act as independent variables, and no other. The increase in efficiency with the use of the generalized eccentric anomaly is explained by that the orbital dynamics along this anomaly is smoothed out not only for the ellipticity of the orbit, but also for the irregular and rather large perturbations from the major planets.

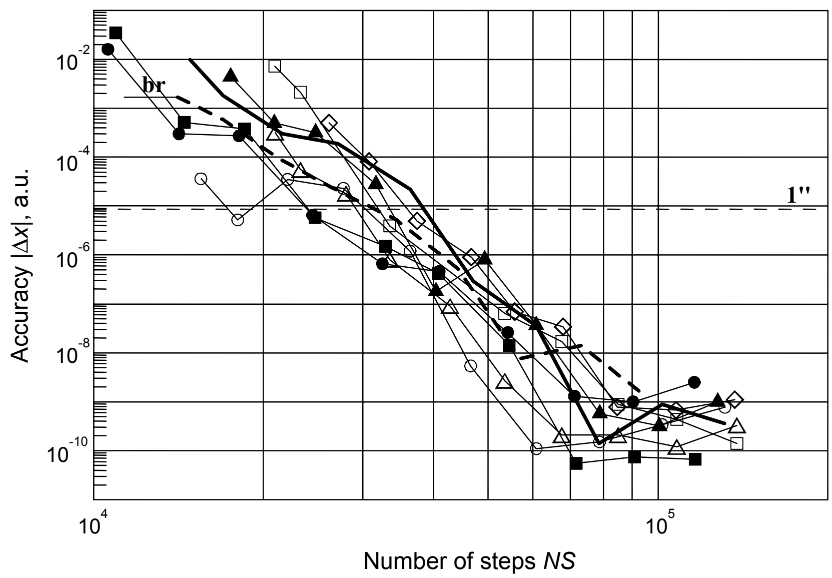

Fig. 9. Ceres.

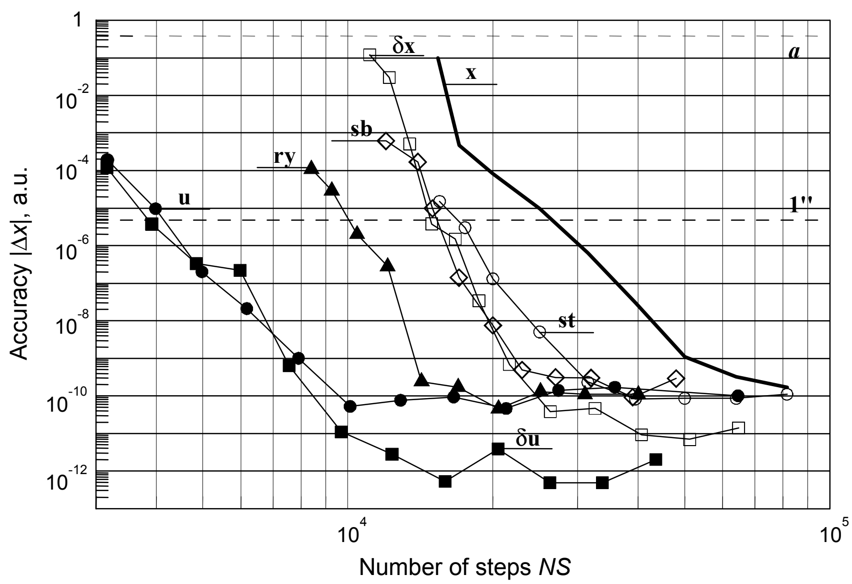

Fig. 10. Mercury.

As practice shows, loss of accuracy of a numerical solution is related in the main to great errors in the time variable due to its nonuniform behavior along an anomaly. Figure 8 shows that introducing a time element τ which behaves linearly in the nonperturbed case allows one to make the numerical integration much more efficient.

For Ceres and Jupiter, the coefficient ν is high enough; therefore, the use of the methods considered is inefficient (Figs. 9 and 11). Moreover, the methods that are used to advantage in the case of Phaeton (KS and SB regularization) (**u**, **δu, sb**) even reduce the efficiency of the numerical integration, for example, for Jupiter. At the same time, it should be noted that in solving the planetary problem the transformation to a barycentric coordinate system (**br**) can substantially weaken the influence of the short-period perturbations from Mercury and thereby increase the integration accuracy by several orders of magnitude (Fig. 11), whereas for the asteroid problem the results are not improved qualitatively.

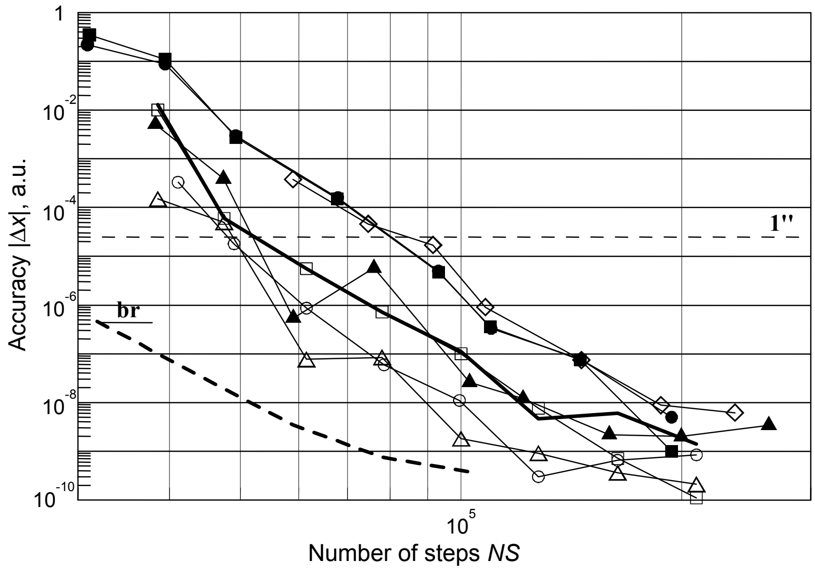

Fig. 11. Jupiter.

In the case of Mercury, the results show (Fig. 10) that for a numerical investigation of its dynamics the equations in KS variables (**u**, **δu**) are very well applicable. Thus, the numerical integration speed can be increased about seven times with the accuracy preserved high enough. As for Phobos, the Encke method (**δu**) allows one to raise the highest possible accuracy by about two orders of magnitude. The low efficiency of the method of variation of arbitrary constants, which is represented by the Roy equations (**ry**), seams to be due to that the orbit of Mercury precesses rather rapidly. This, in turn, leads to rapid variations of the momentum vector $\boldsymbol{c}$, which, probably, is integrated with insufficient accuracy.

Finally, as to the Encke method (**δx**), which has not yet received our comments, the results have shown that it turned out least efficient among all methods considered.

CONCLUSION

Thus, the methods of special perturbation theory are appropriate for use only in those problems where short-period perturbing forces are absent, or, if any, only if $\nu < 1$. In the absence of short-period perturbations, the KS regularization (**uδ**) and the Encke method in KS variables (**δu**) are highly efficient. Besides, for numerical investigation of the dynamics of close satellites the method of variation of variables in Roy's interpretation (**ry**) can also be recommended.

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